ECE 205 “Electrical and Electronics Circuits”

Spring 2024 – LECTURE 12

MWF – 12:00pm

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Lecture 12 – Summary

Learning Objectives

1. Understand transient behavior of circuits
2. Introduce the capacitor circuit element
3. Analysis of circuits containing resistors and capacitors

Transient analysis

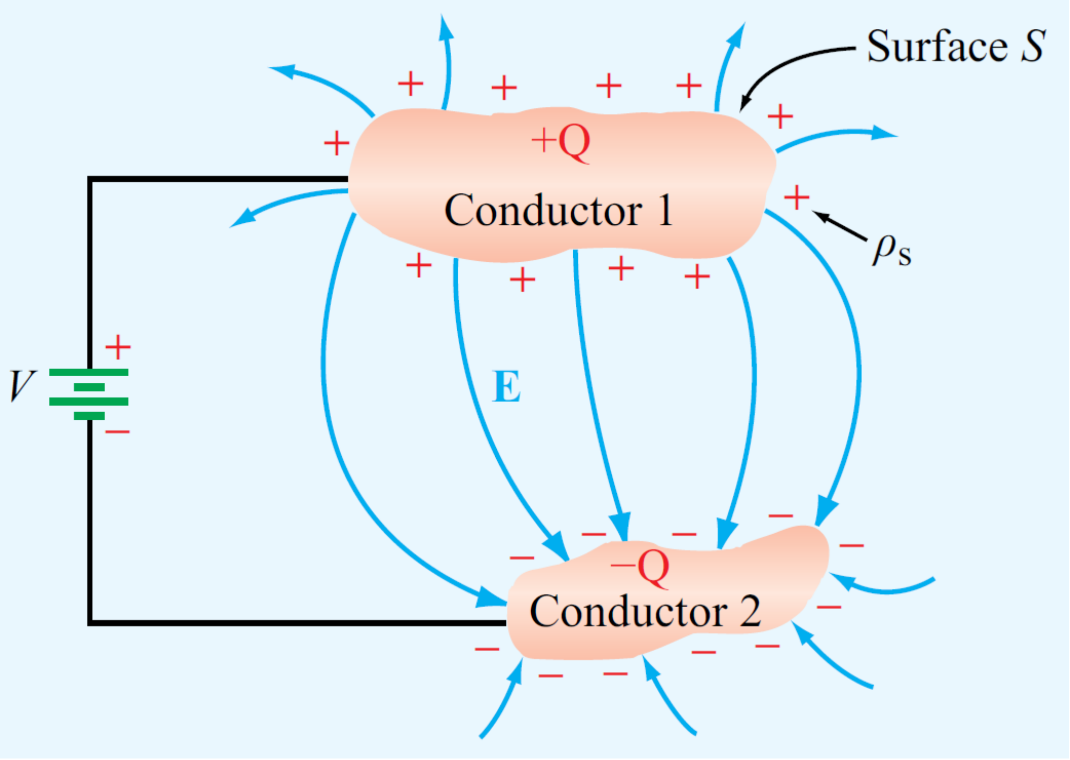
Until now, we have considered circuits in Direct Current (DC) regime.

Even in the case of DC sources, a circuit experiences a time-dependent behavior when, for instance, a switch is opened or closed.

In the "transient" regime, we introduce new circuit components: the capacitor and the inductor.

Capacitors

The **capacitor** is a two terminal device consisting of two separate metallic masses in close proximity which, under the application of a voltage, can store charges of opposite polarity with establishment of an electric field in the gap between them.



The capacitance of a two-conductor configuration is defined as

$$C = \frac{Q}{V} \qquad (\mathrm{C/V\ or\ F})$$

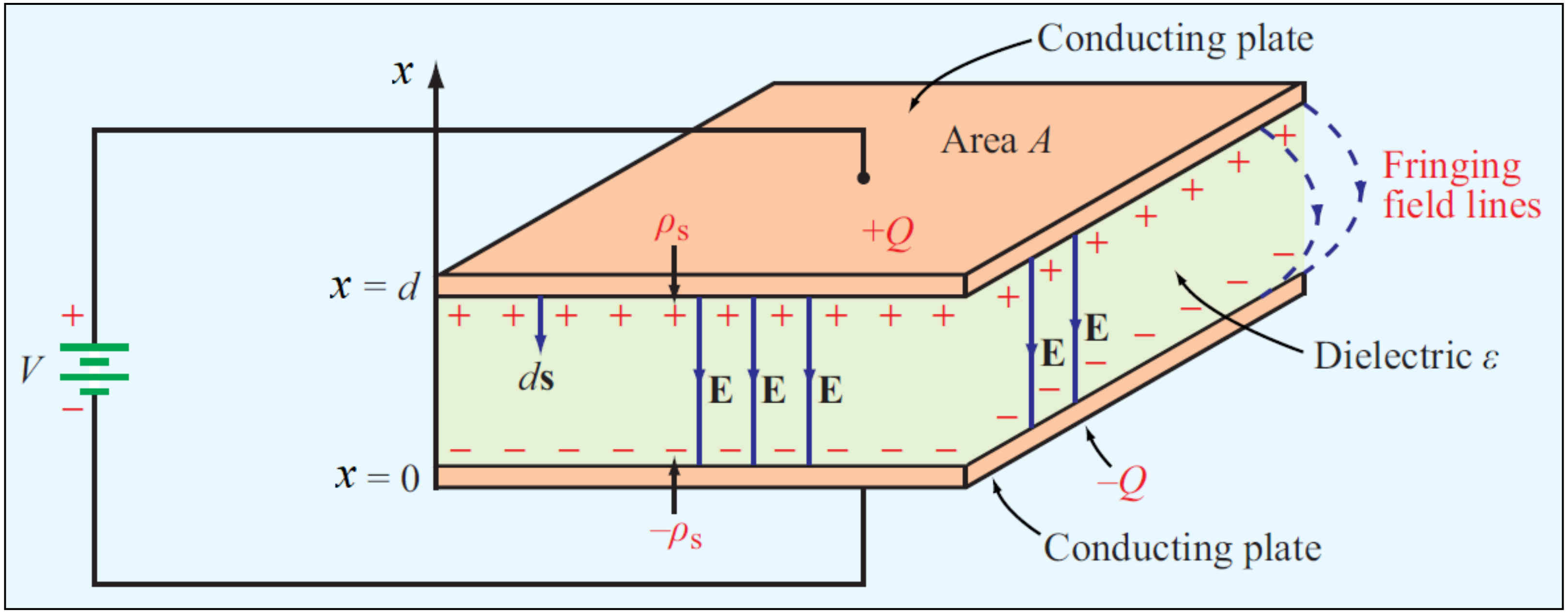
C/V → Coulomb/Volt

F → Farad

The capacitance C is independent of the electric field magnitude. It depends only on geometry (size, shape, positions) and on permittivity of the insulator (linear capacitor).

The simplest form is the parallel plate capacitor

Electric field $\mathbf{E}(x) = \text{constant} \longrightarrow V = \mathbf{E}\, d$

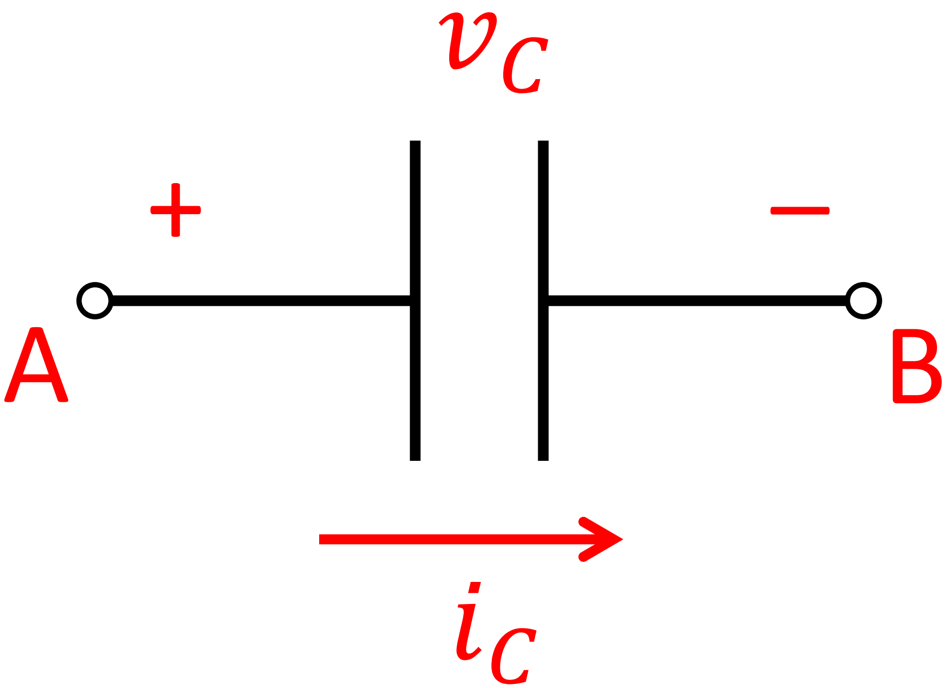


Capacitance of the parallel plate capacitor

$$C = \frac{Q}{V} = \frac{\epsilon A}{d}$$

ε = dielectric constant or permittivity (F/m)
A = Area of plates (m^2)
D = distance between plates (m)

Circuit symbol of a capacitor



Relationship between current and voltage

$$i_C = C\frac{dv_C(t)}{dt}$$

$$i_C = C\frac{dv_C(t)}{dt}$$

- The capacitor is a natural "differentiator" of voltage.
- When

$$\frac{dv_C}{dt} = 0 \quad \rightarrow \quad i_C = 0$$

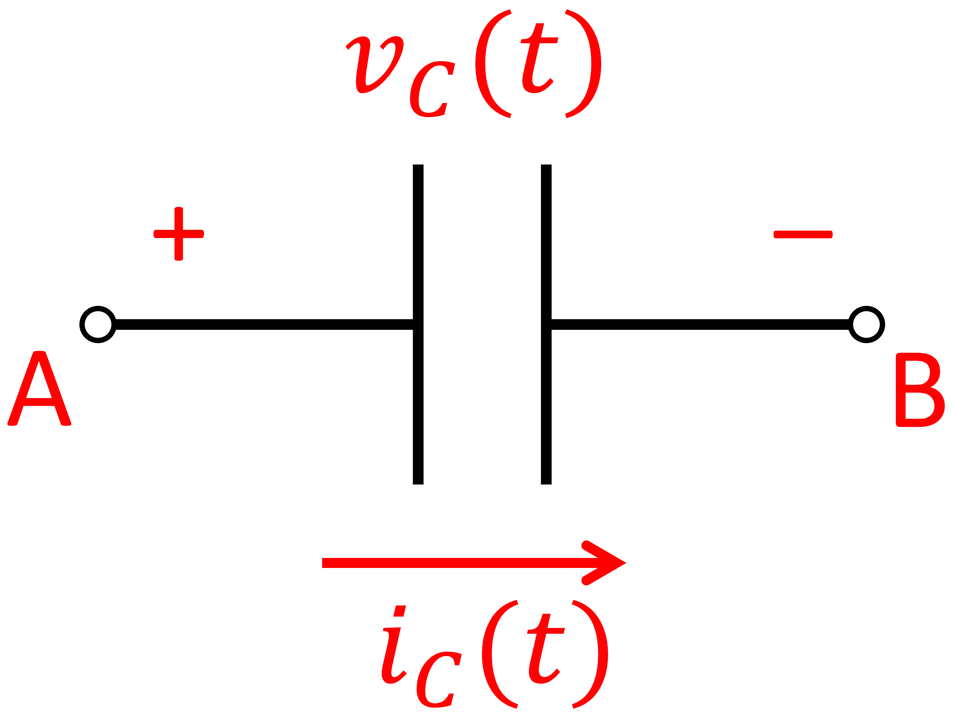
the capacitor behaves like an open circuit

- The voltage across a capacitor cannot change instantaneously ($dt \rightarrow 0$ would imply $i_C \rightarrow \infty$)

Time-behavior of capacitor

From the definition of capacitance

$$C \equiv \frac{Q}{V}$$

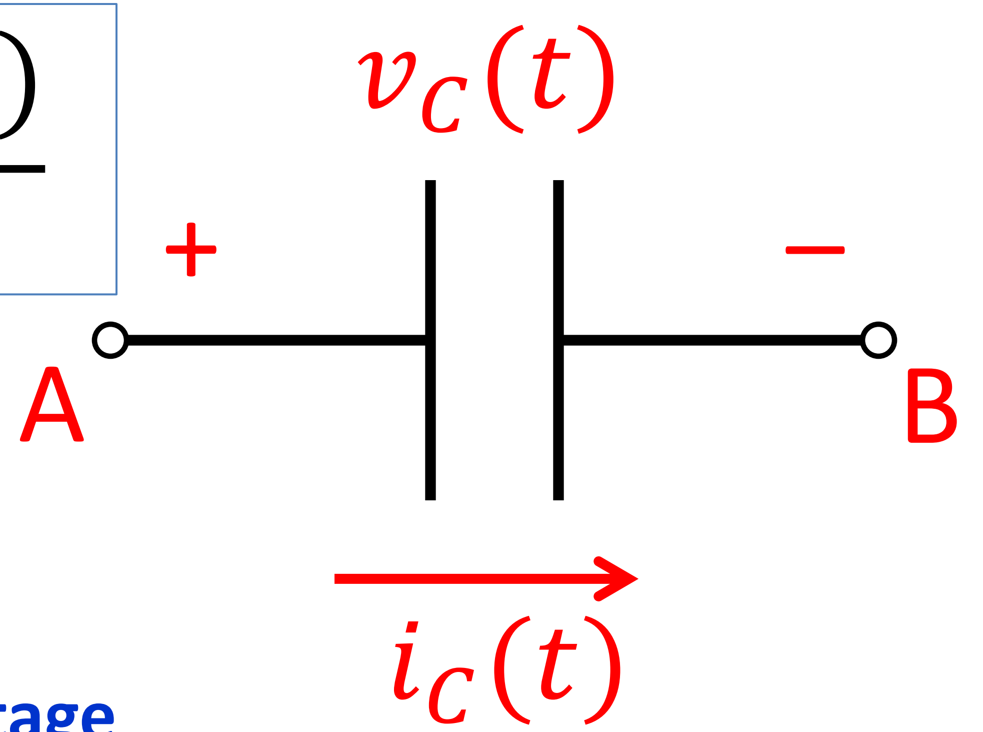


$$q(t) = C v_c(t)$$

$$\frac{d}{dt} q(t) = i_C(t) = C \frac{d}{dt} v_C(t)$$

Time-behavior of capacitor

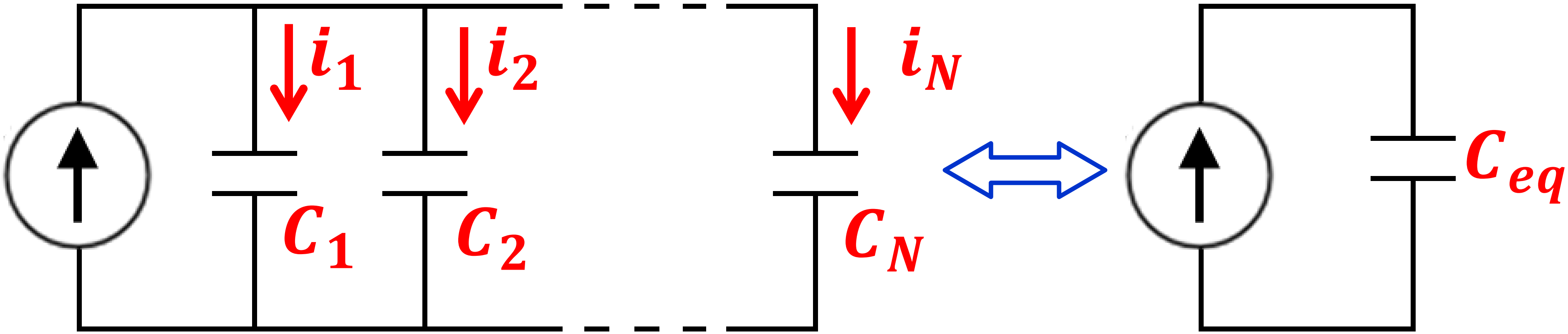
$$i_C(t) = C\frac{dv_C(t)}{dt}$$



Change of capacitor voltage in time interval $[t_0, t]$

$$v_C(t) = v_C(t_0) + \frac{1}{C}\int_{t_0}^{t} i_C(t')\,dt'$$

Parallel of N capacitors



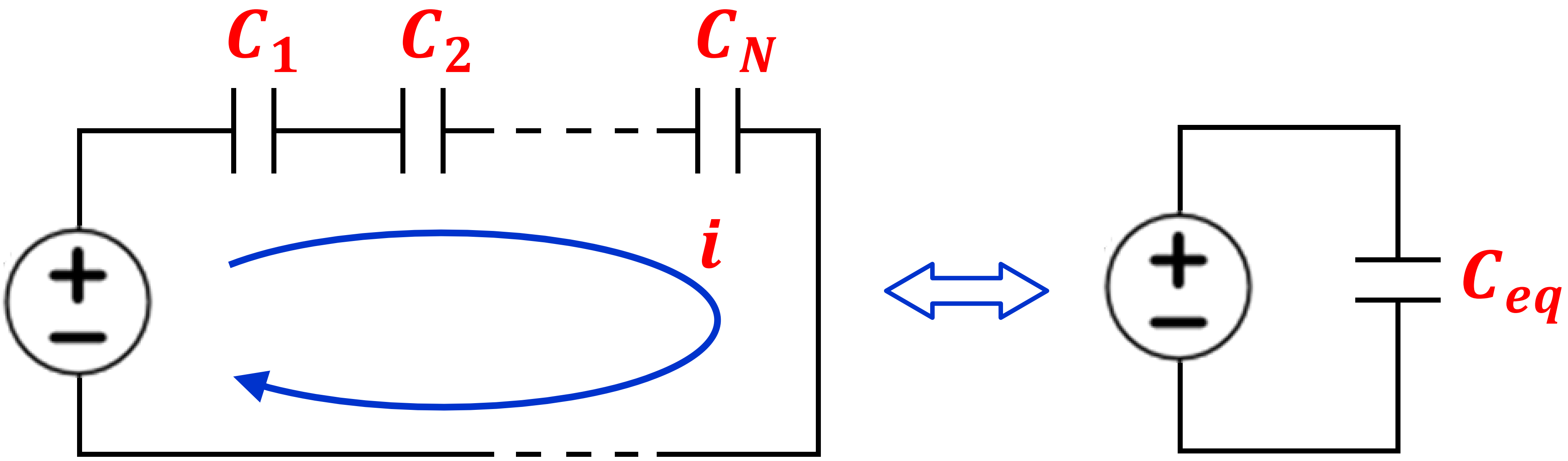
Equivalent Capacitance

$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N$$

$$C = \frac{\epsilon A}{d}$$

It is as if we are increasing the area by placing capacitors in parallel

Series of N capacitors



Equivalent Capacitance

$$C_{eq} = \left[\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right]^{-1}$$

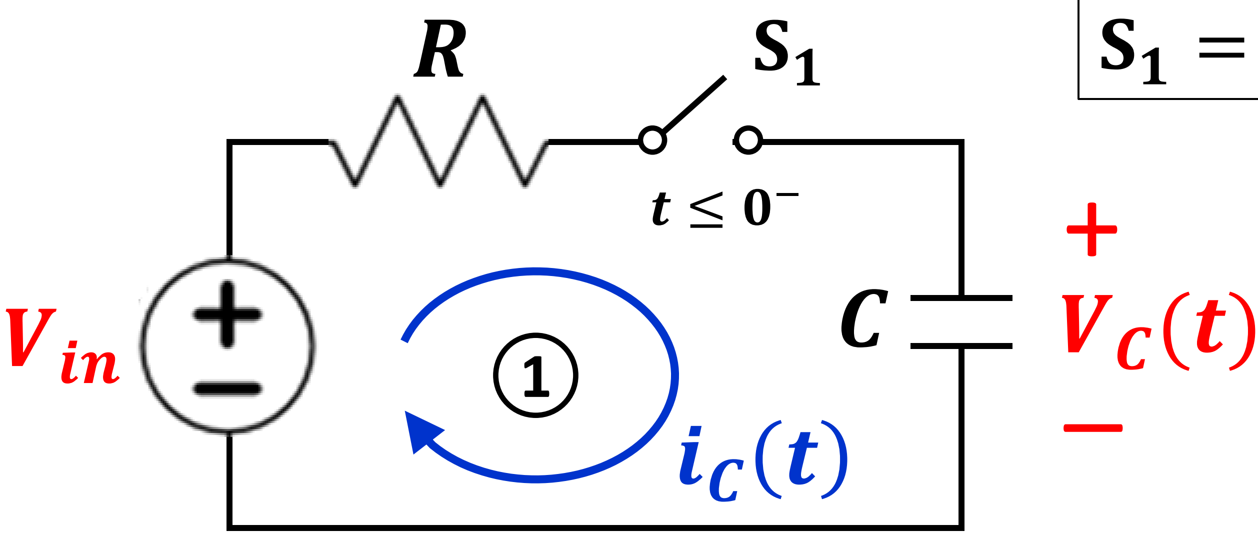
$$C = \frac{\epsilon A}{d}$$

It is as if we are increasing the distance between the first and the last plate by placing capacitors in series

RC Circuits – Transient Analysis

Response to “step input”

The transient analysis involves computing the time evolution of voltage and current after closing the switch.

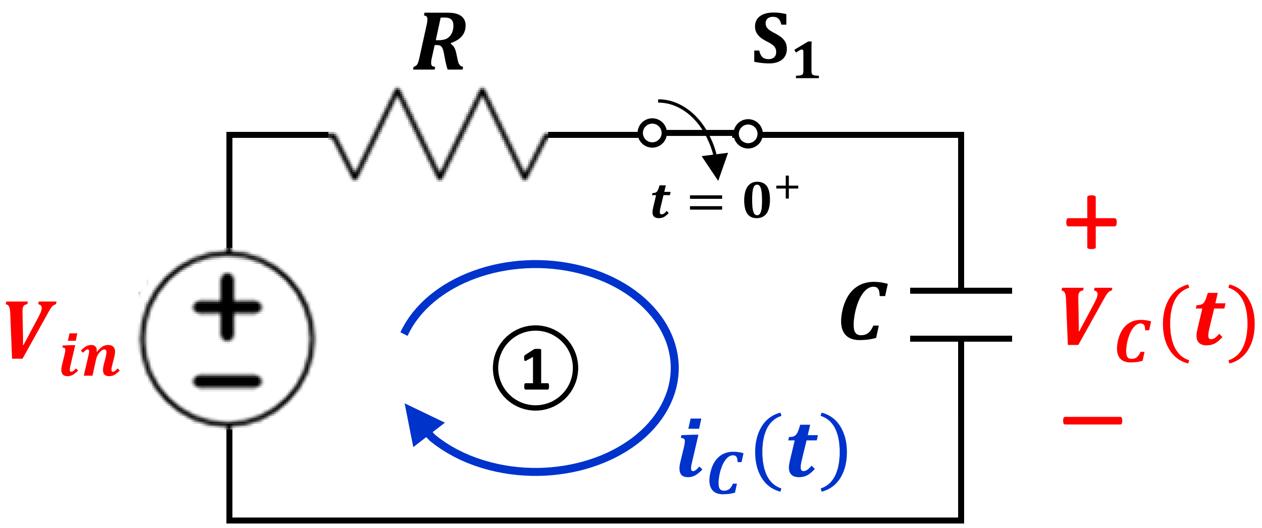


S_1 = switch

V_{in} = constant voltage source (it does not change in time)

The switch is open until $t = 0^-$ and it closes at $t = 0^+$

$$V_C(t = 0^-) = V_C(t = 0^+)$$



As said earlier, voltage across a capacitor cannot change instantaneously, otherwise

$$i_C = C\frac{dV_C(t)}{dt}$$

would require infinite current.

Transient Timeline

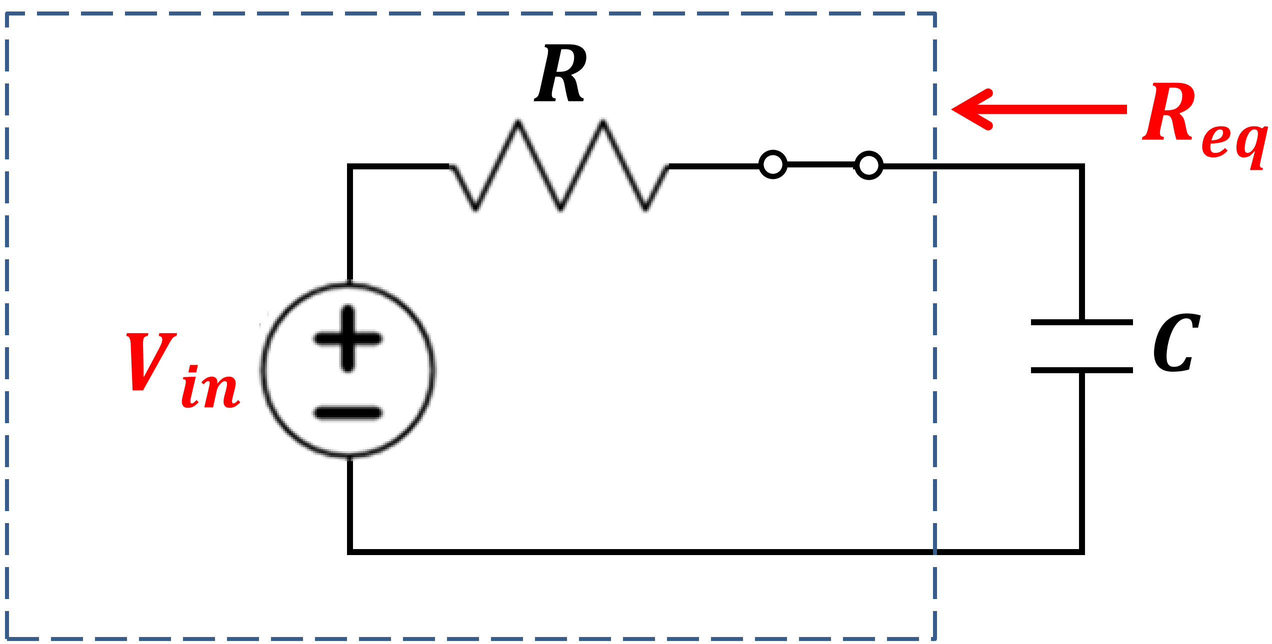
$(t = 0^+)$ initial state

$(t = \tau)$ RC time constant

$(t \rightarrow \infty)$ steady state

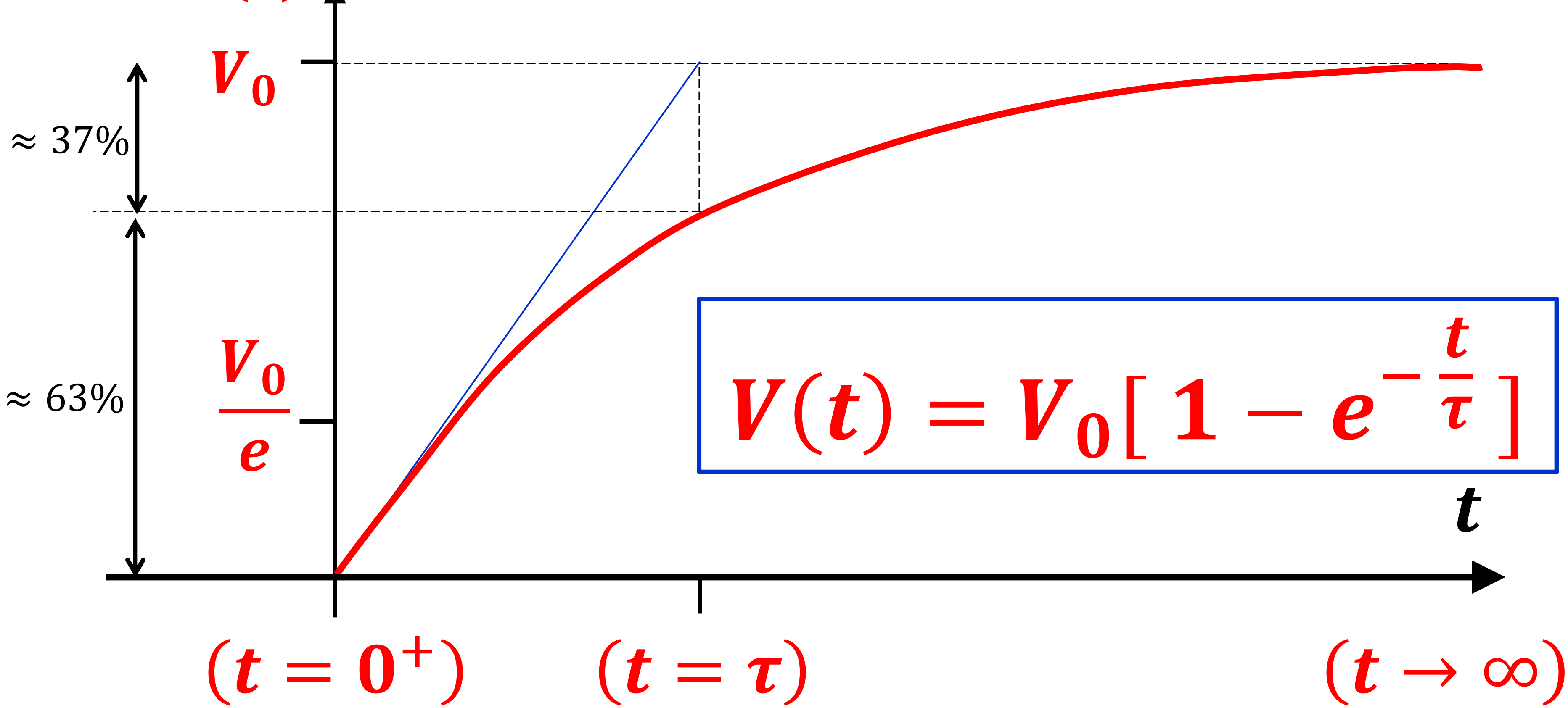
time constant

$$\tau = R_{eq}\, C$$



seen by the capacitor

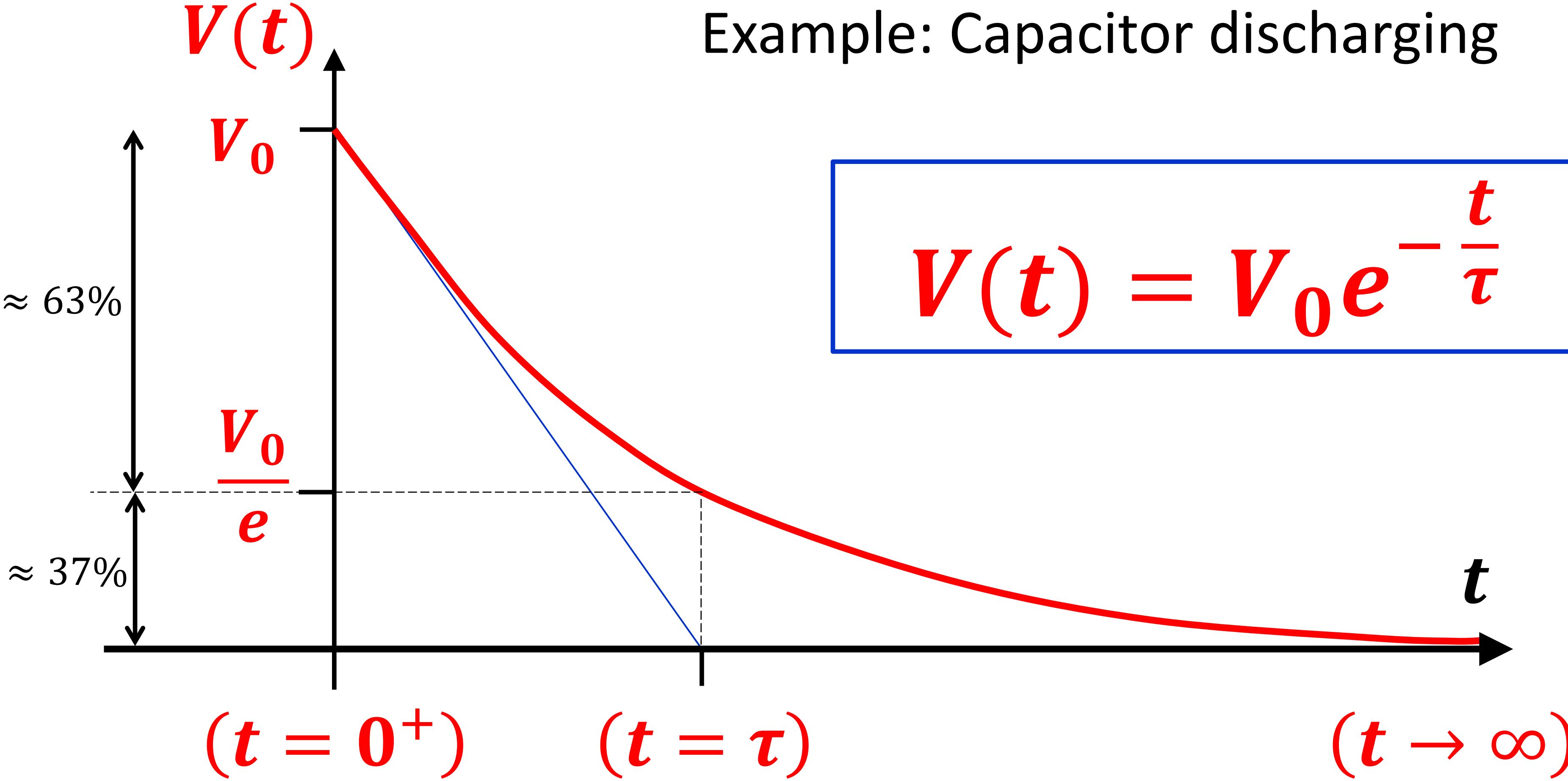
Example: Capacitor charging



$$V(t) = V_0\left[\,1 - e^{-\frac{t}{\tau}}\,\right]$$

$$\tau = R_{eq}\, C$$

Example: Capacitor discharging



$$V(t) = V_0 e^{-\frac{t}{\tau}}$$

$$\tau = R_{eq}\,C$$

Capacitors as "energy" devices

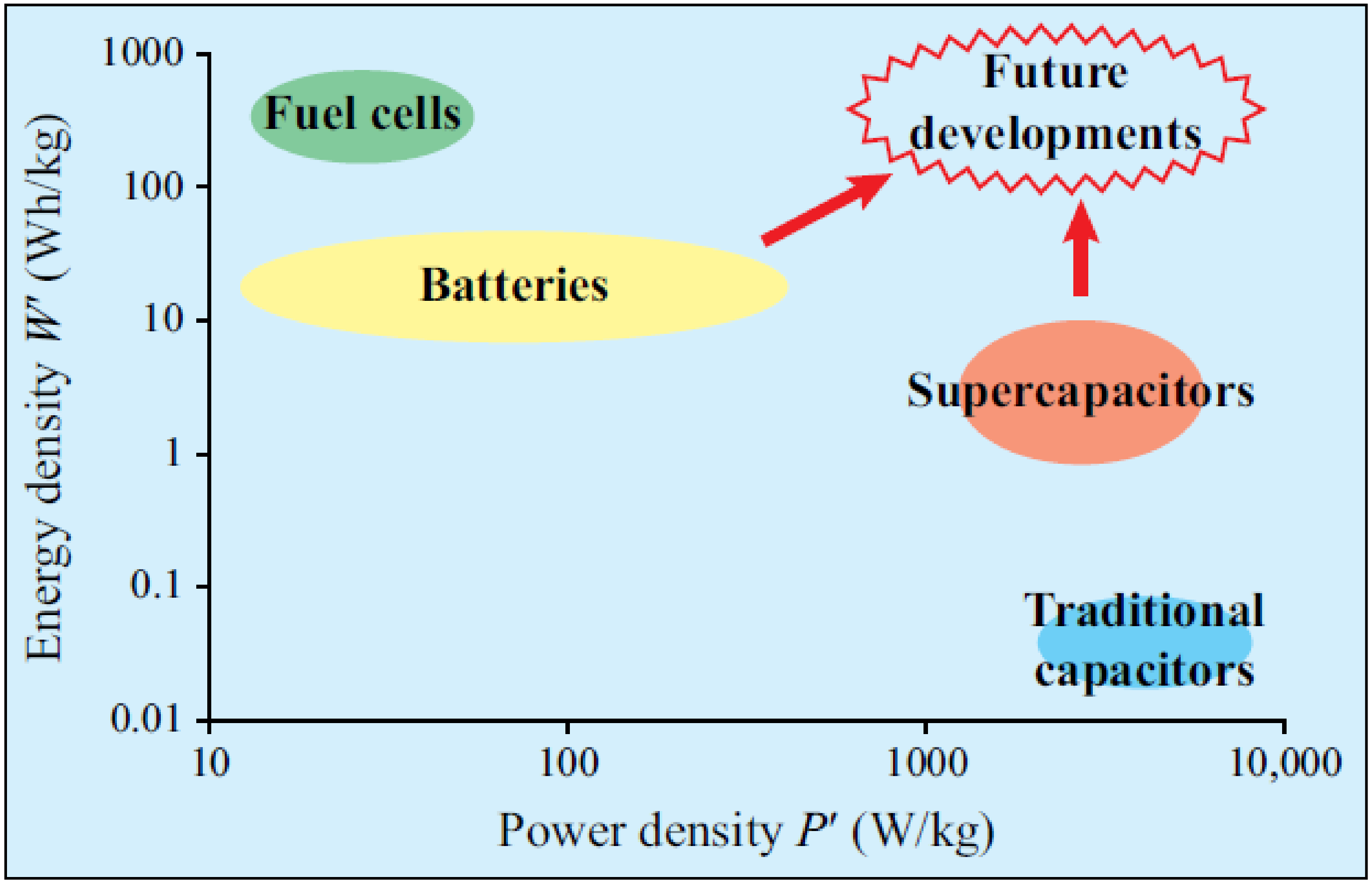
Capacitors store electric energy when charged and release energy when discharged.

Batteries store much more energy than capacitors but capacitors release energy much faster than batteries. Special devices called "supercapacitors" are used to supplement batteries when short high energy bursts are needed.

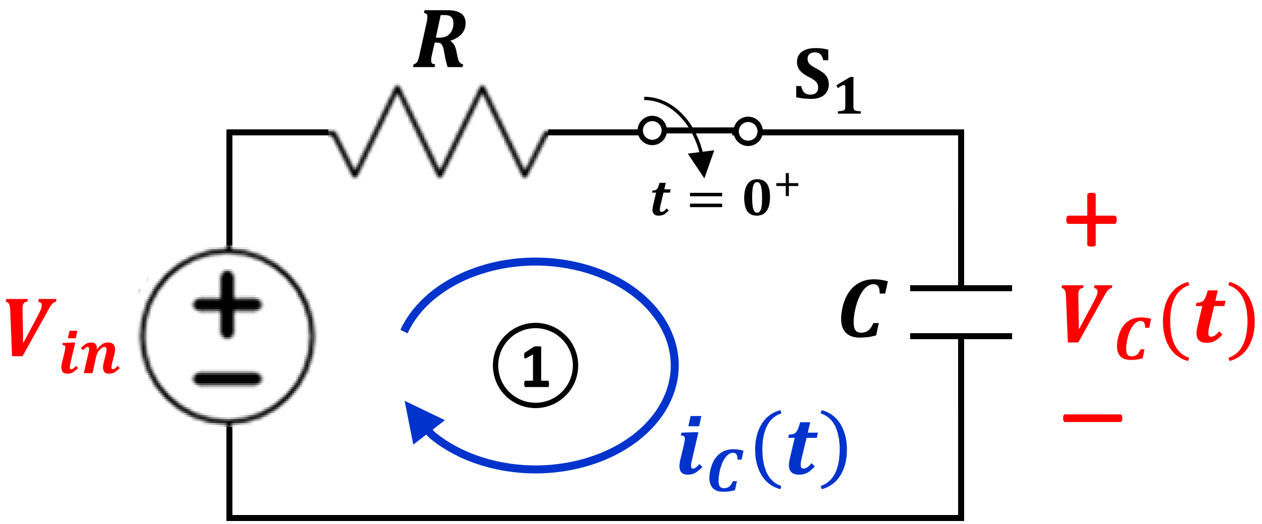
Supercapacitors are designed to have a very large effective surface and the minimum possible (safe) distance between plates, so that they can store more energy than traditional circuit components.

Energy storage devices

Feature	Traditional Capacitor	Supercapacitor	Battery
Energy density W' (Wh/kg)	$\sim 10^{-2}$	1 to 10	5 to 150
Power density P' (W/kg)	1,000 to 10,000	1,000 to 5,000	10 to 500
Charge and discharge rate T	10^{-3} sec	$\sim$ 1 sec to 1 min	$\sim$ 1 to 5 hrs
Cycle life N_c	∞	$\sim 10^6$	$\sim 10^3$



RC circuit analysis for constant input voltage



KVL loop ①

$$R\, i_C(t) + V_C(t) - V_{in} = 0$$

Since

$$i_C = C\frac{dV_C(t)}{dt}$$

$$RC\frac{dV_C(t)}{dt} + V_C(t) - V_{in} = 0$$

$$\frac{dV_C(t)}{dt} + \frac{(V_C(t) - V_{in})}{RC} = 0$$

First order ordinary differential equation in time, for one energy storage element. N-th order ODE for N energy storage elements.

General solution has the form

$$V_C(t) = K_1 e^{-\alpha t} + K_2 \qquad [\mathrm{V}]$$

Constants K_1 and K_2 obtained from initial and final conditions

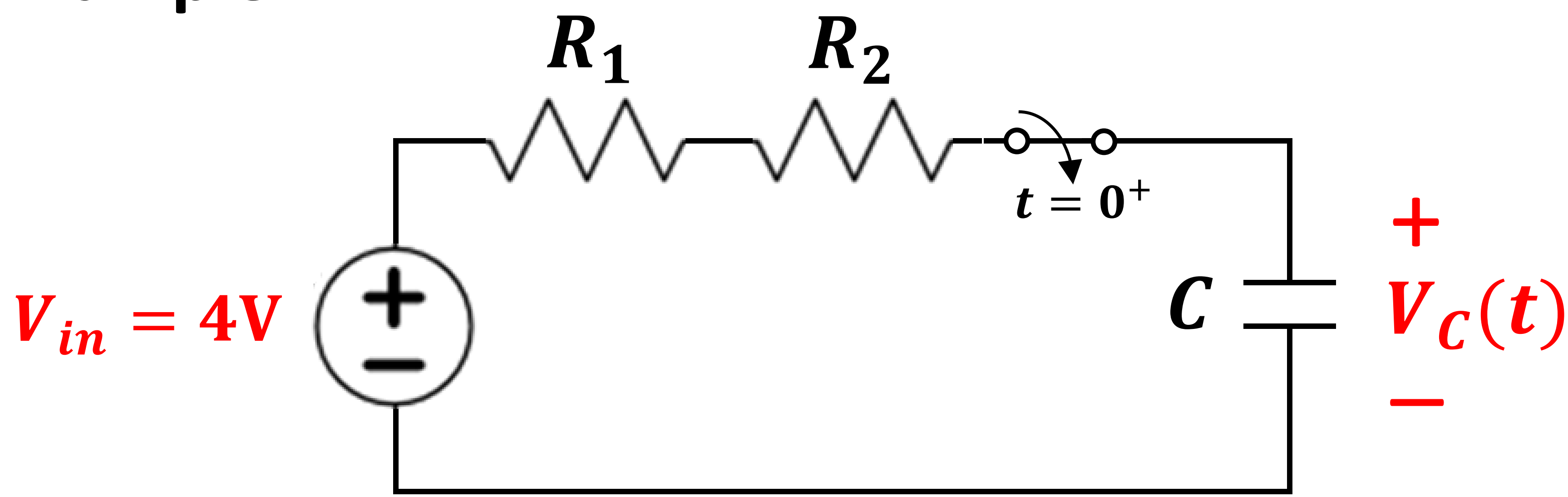
$$V_C(t \to \infty) = K_2$$

$$V_C(t \to 0^+) = K_1 + K_2$$

and

$$\alpha = \frac{1}{\tau} = \frac{1}{R_{eq}C}$$

Example 1



$$R_1 = R_2 = 2k\Omega$$

$$C = 0.25\ \mu F$$

$$V_C(t = 0^-) = 6\text{V}$$

Switch closes at $t = 0^+$

$$\boxed{V_C(t) = K_1 e^{-\alpha t} + K_2}$$

Example 1

$$V_C(t) = K_1 e^{-\alpha t} + K_2$$

Step ① **Find constants K_1 and K_2**

$$V_C(t = 0^-) = V_C(t = 0^+) = 6\text{V}$$

$$V_C(t = 0^+) = 6\text{V} = K_1 + K_2$$

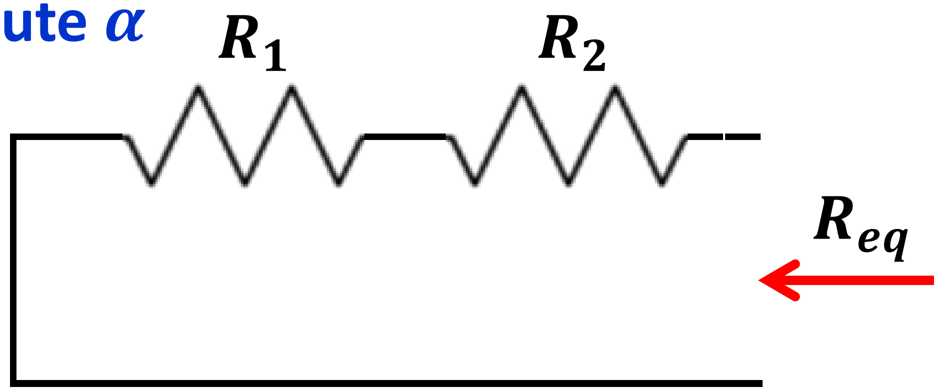
(No current flows) $V_C(t \to \infty) = V_{in} = K_2 = 4\text{V}$

⇨ $K_1 = 2\text{V}$

$K_2 = 4\text{V}$

Example 1

Step ② **Compute α**



$$R_{eq} = R_1 + R_2 = 4\ \mathrm{k\Omega}$$

$$\tau = R_{eq}C = 4\ \mathrm{k\Omega} \times 0.25\ \mathrm{\mu F} = 10^{-3}\ \mathrm{s}$$

$$\alpha = \frac{1}{\tau} = 10^{3}\ \mathrm{s}^{-1}$$

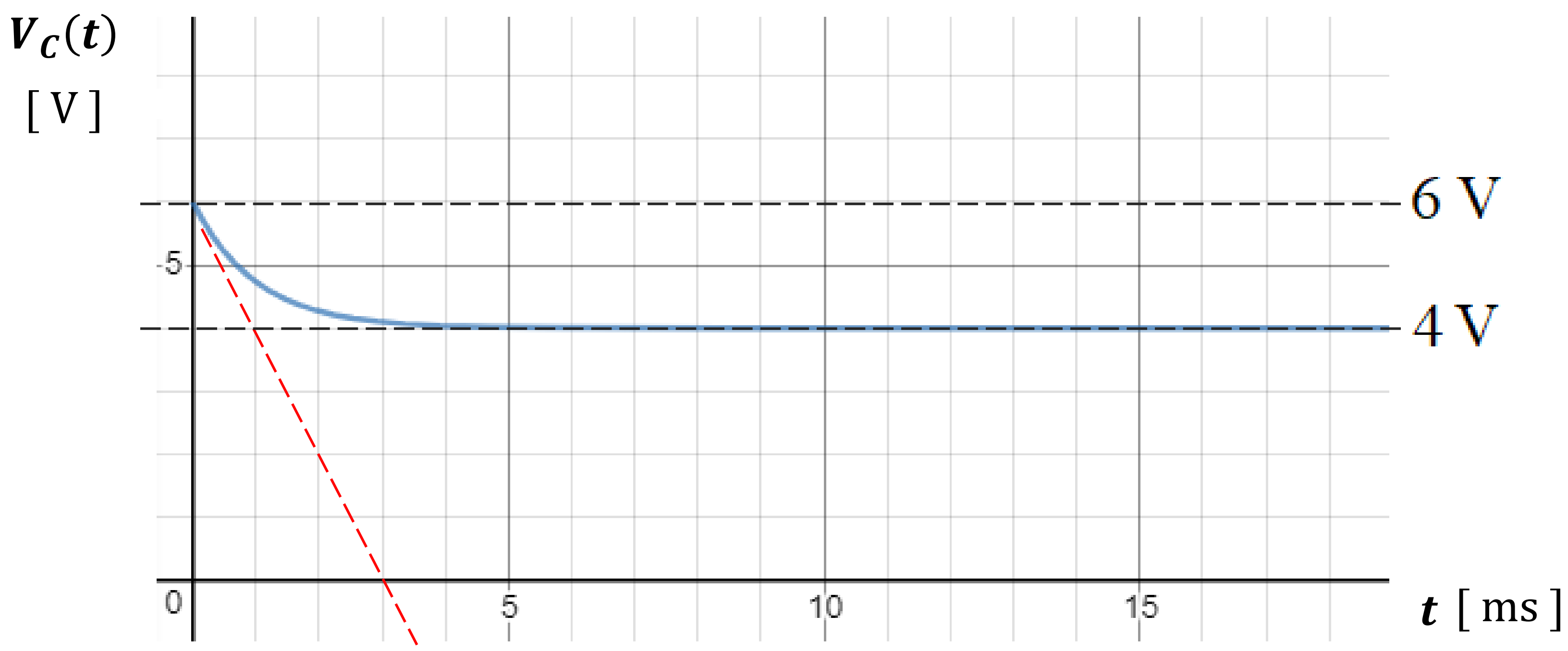
Step ③ **Time-dependent voltage solution**

$$V_C(t) = K_1 e^{-\alpha t} + K_2 = 2\ e^{-10^3 t} + 4\ \ [\mathrm{V}]$$

Example 1

Time-dependent voltage solution

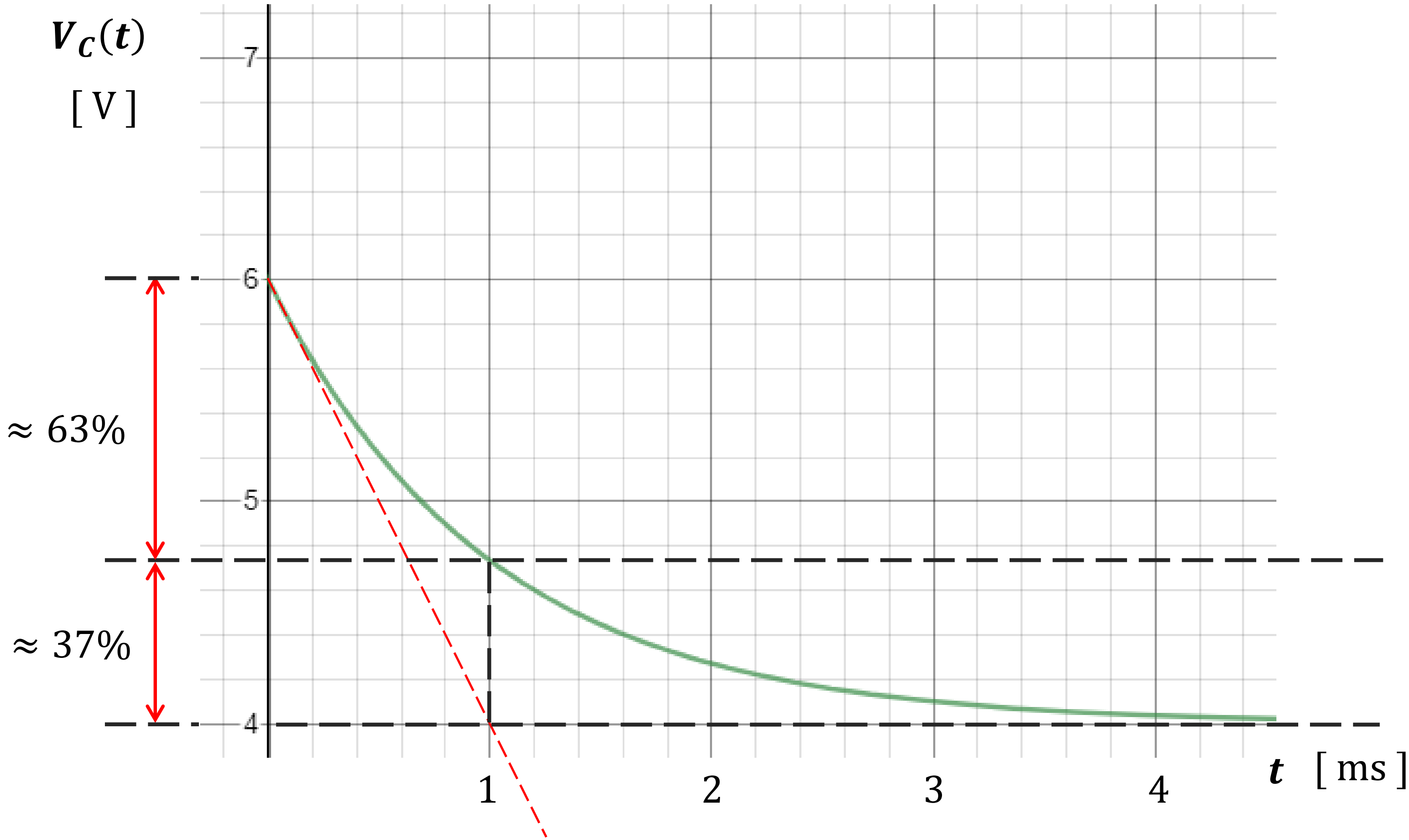
$$V_C(t) = 2\, e^{-10^3 t} + 4 \;\; [\mathrm{V}]$$



Example 1

Time-dependent voltage solution

$$V_C(t) = 2\, e^{-10^3 t} + 4 \;\; [\mathrm{V}]$$



Example 1

Time-dependent current solution

$$i_C(t) = C\frac{dV(t)}{dt} = C\frac{d}{dt}\left[2\,e^{-10^3 t} + 4\right]\ [\mathrm{A}]$$

$$= \overbrace{0.25\times10^{-6}}\times\overbrace{\left(-2\times10^3\times e^{-10^3 t}\right)}\ [\mathrm{A}]$$

$$= -0.5\,e^{-10^3 t}\ [\mathrm{mA}]$$