ECE 205 “Electrical and Electronics Circuits”

Spring 2024 – LECTURE 15

MWF – 12:00pm

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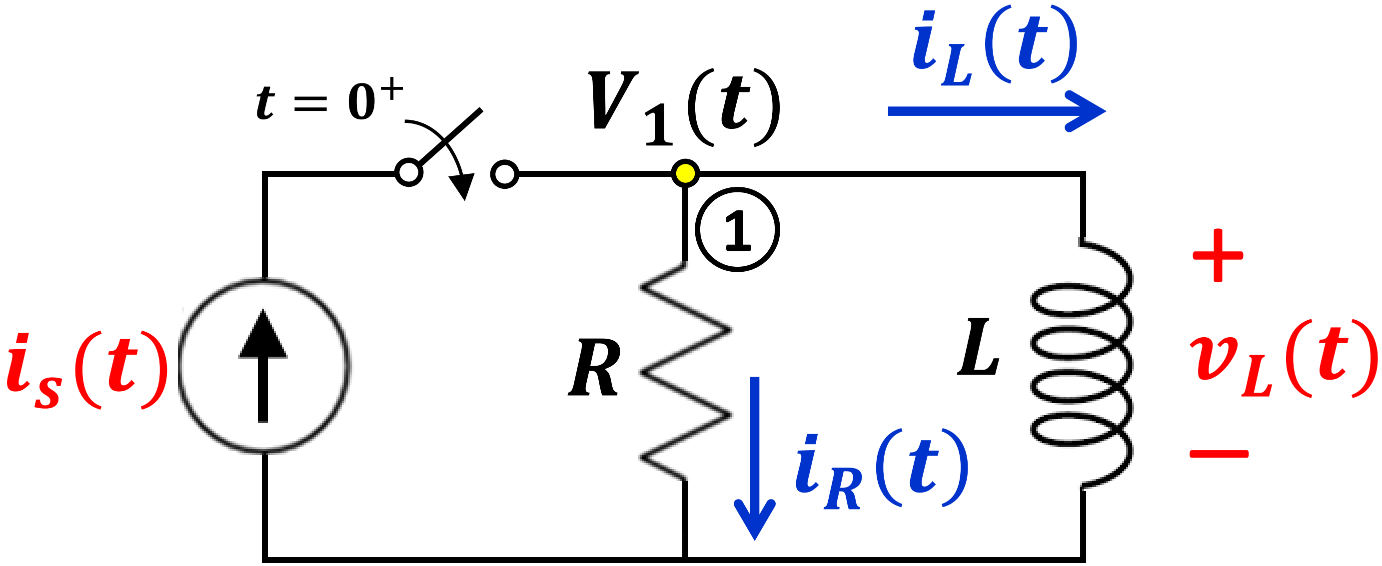
2062 ECE Building

Lecture 15 – Summary

Learning Objectives

1. Finish introductory discussion on RL circuits
2. Power considerations
3. Detailed analysis of RC and RL circuits with time-dependent sources
4. Solution of circuit differential equations

RL Circuits – Transient Analysis
Response to "step input"

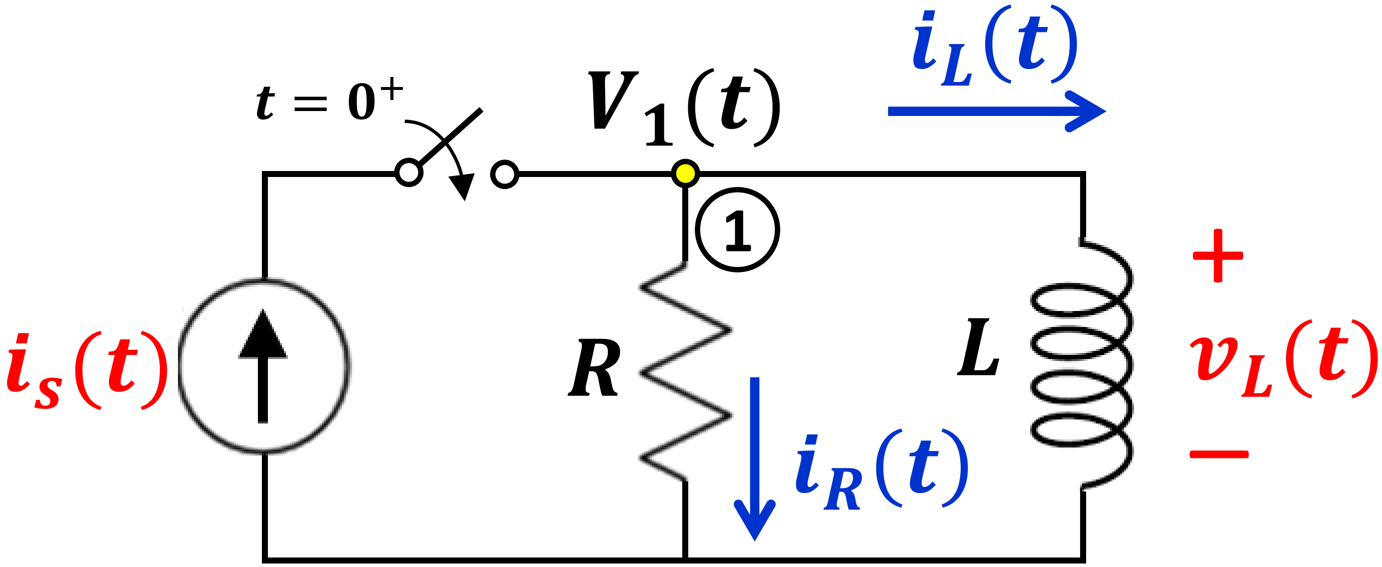


Write KCL at node ①

$$i_s(t) = i_R(t) + i_L(t) = \frac{V_1(t)}{R} + i_L(t)$$

$$V_1(t) = V_L(t) = L\frac{d}{dt}\,i_L(t)$$

$$i_S(t) = \frac{L}{R}\,\frac{d}{dt}i_L(t) + i_L(t)$$



$$i_S(t) = \frac{L}{R}\frac{d}{dt}i_L(t) + i_L(t)$$

$$\frac{d}{dt}i_L(t) + i_L(t)\frac{R}{L} = i_S(t)\frac{R}{L}$$

Equation 1

$$\frac{d}{dt} i_L(t) + i_L(t)\frac{R}{L} = i_S(t)\frac{R}{L}$$

Equation 1

This equation has the same mathematical form of the Ordinary Differential Equation for the voltage in a series RC circuit, considering a constant current source.

General Solution

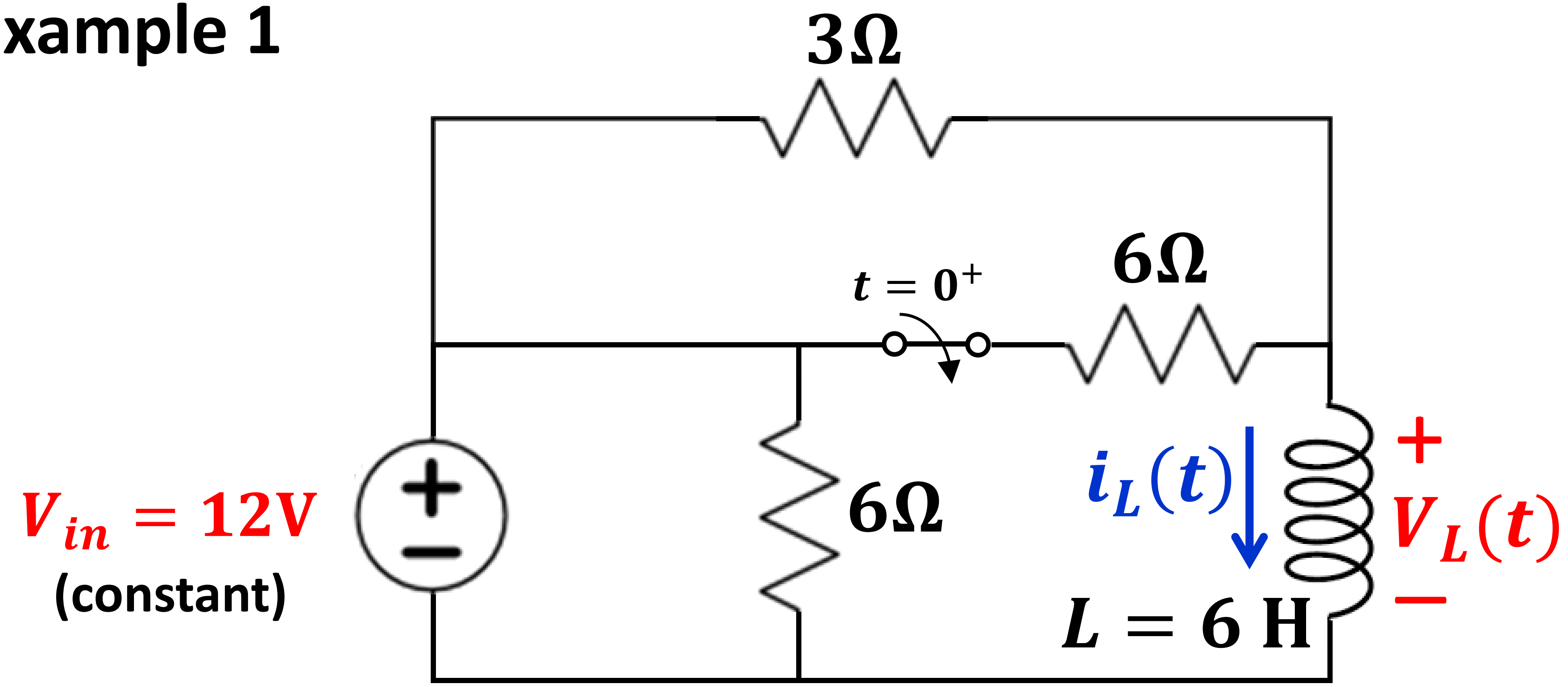
$$i_L(t) = K_1 e^{-\alpha t} + K_2 \quad [\mathrm{A}]$$

$$i_L(t \to \infty) = K_2$$

$$i_L(t \to 0^+) = i_L(t \to 0^-) = K_1 + K_2$$

$$\tau = \frac{L}{R_{eq}} \qquad \alpha = \frac{1}{\tau} = \frac{R_{eq}}{L}$$

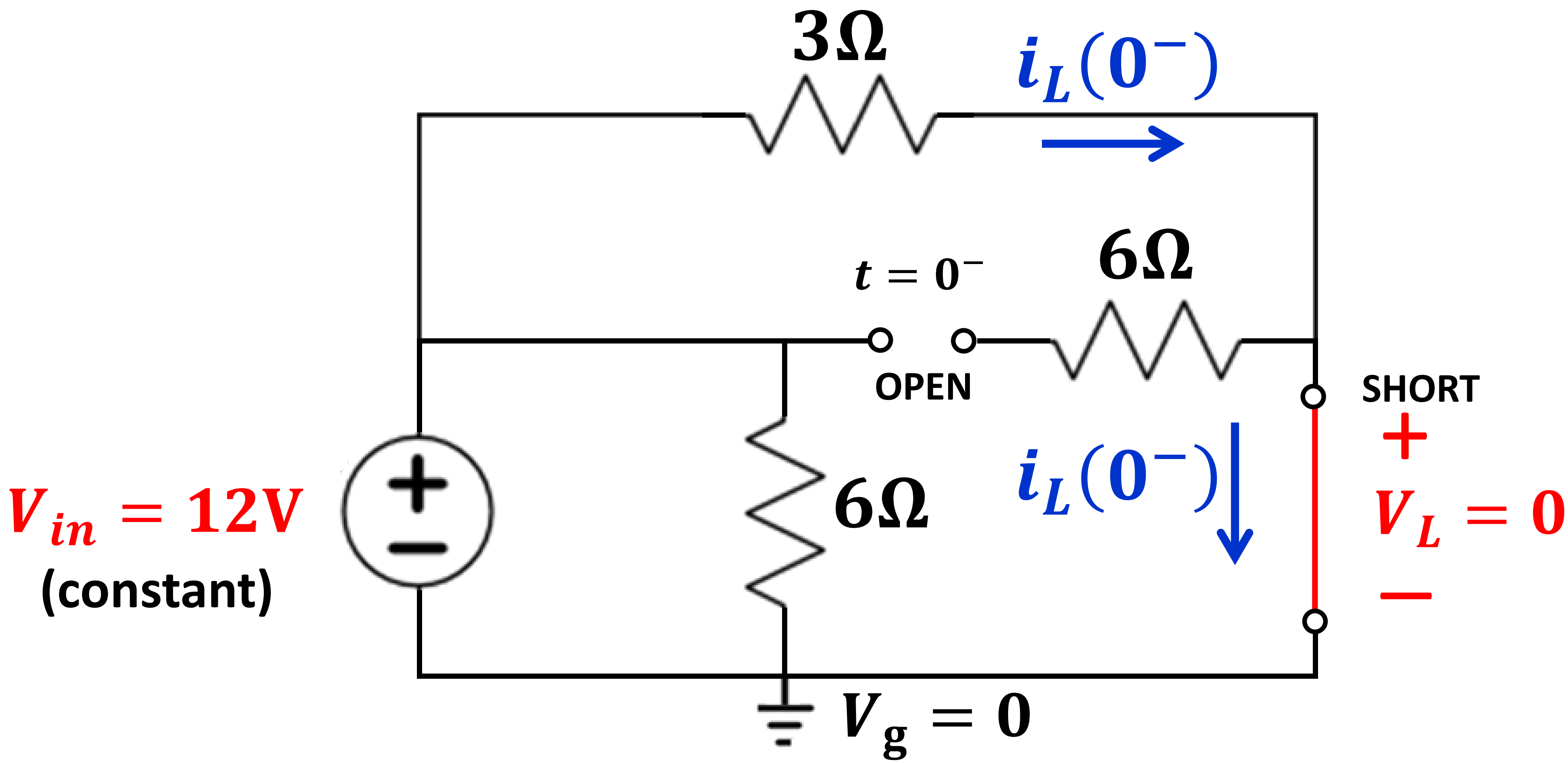
Example 1



Find $i_L(t)$

Switch closes at $t = 0^+$

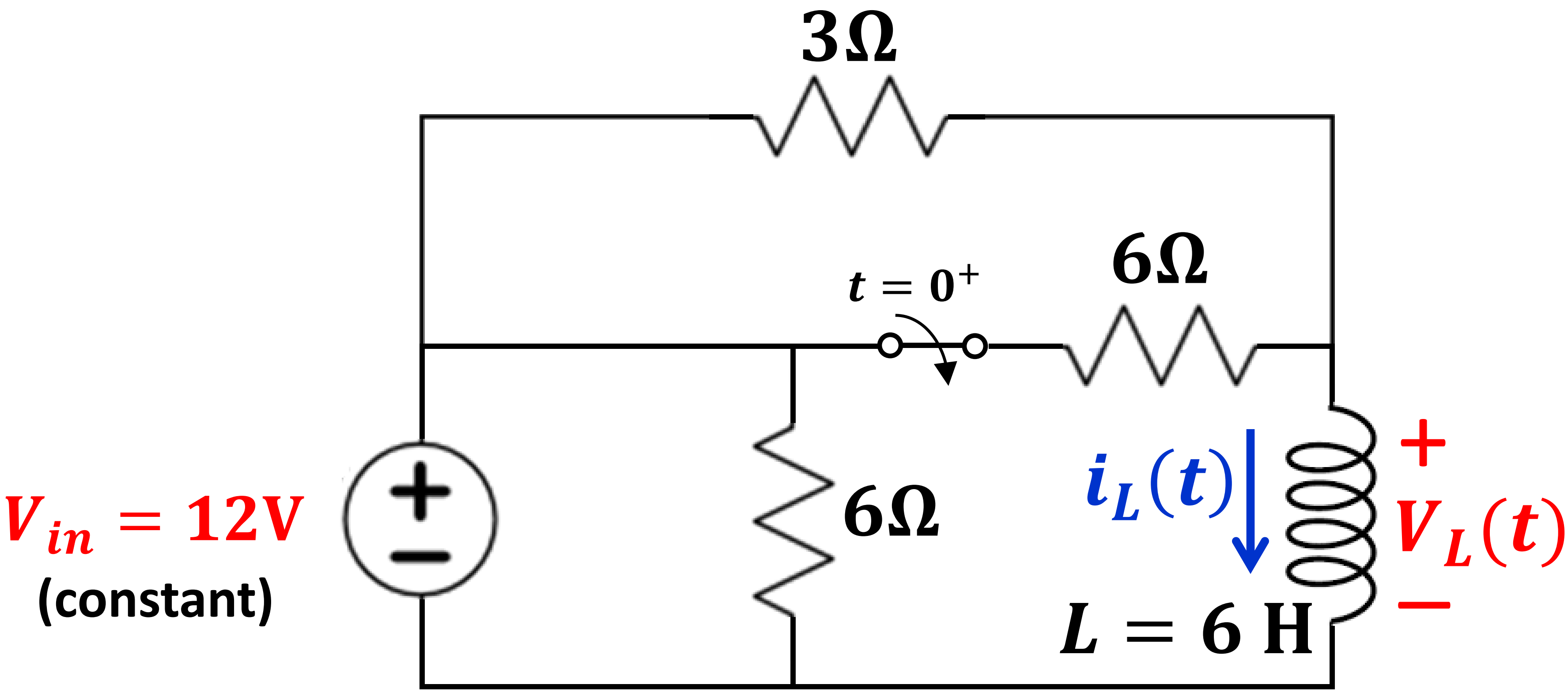
$$i_L(t) = K_1 e^{-\alpha t} + K_2 \quad [\text{A}]$$



Step ① Find $i_L(0^-)$ and $V_L(0^-)$ before the switch is closed

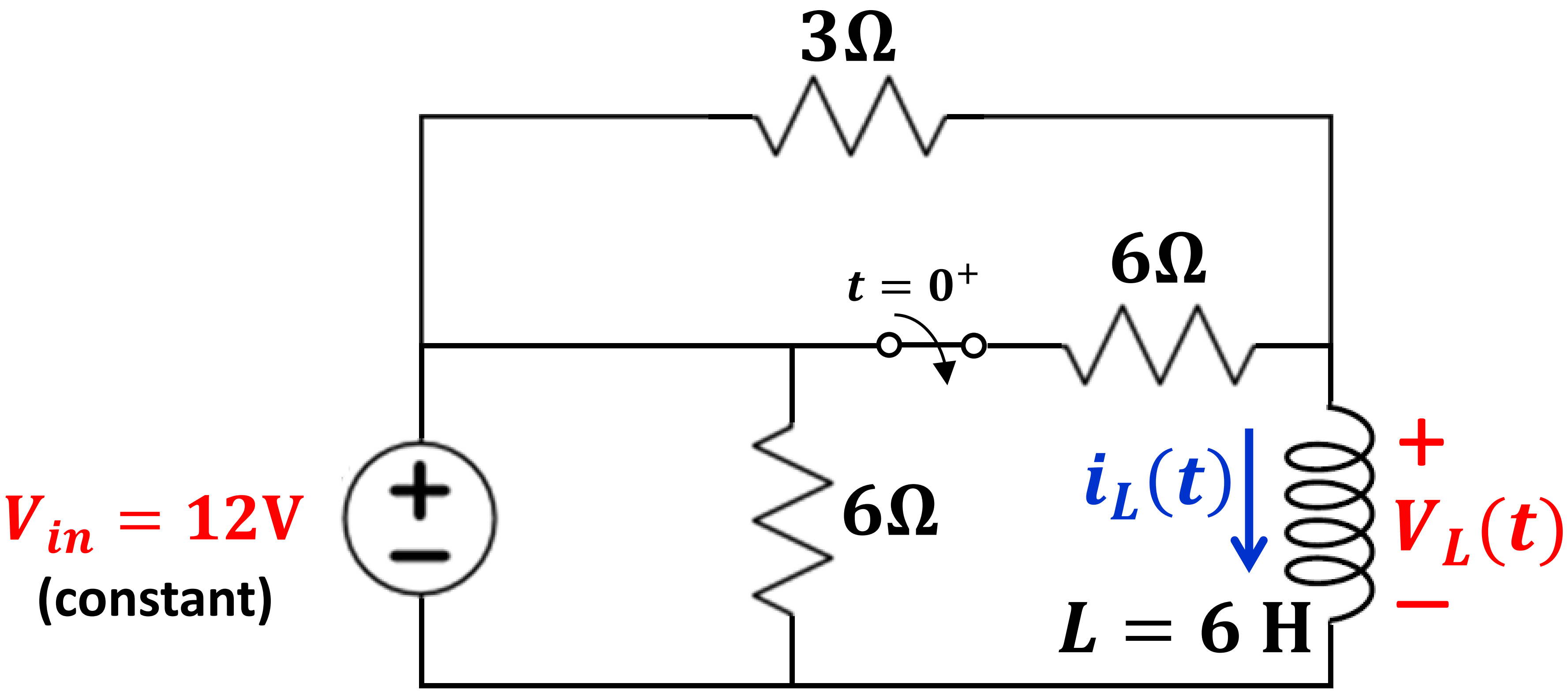
$$i_L(0^-) = \frac{12 - 0}{3} = 4\ A$$

$$V_L(0^-) = 0\ V$$

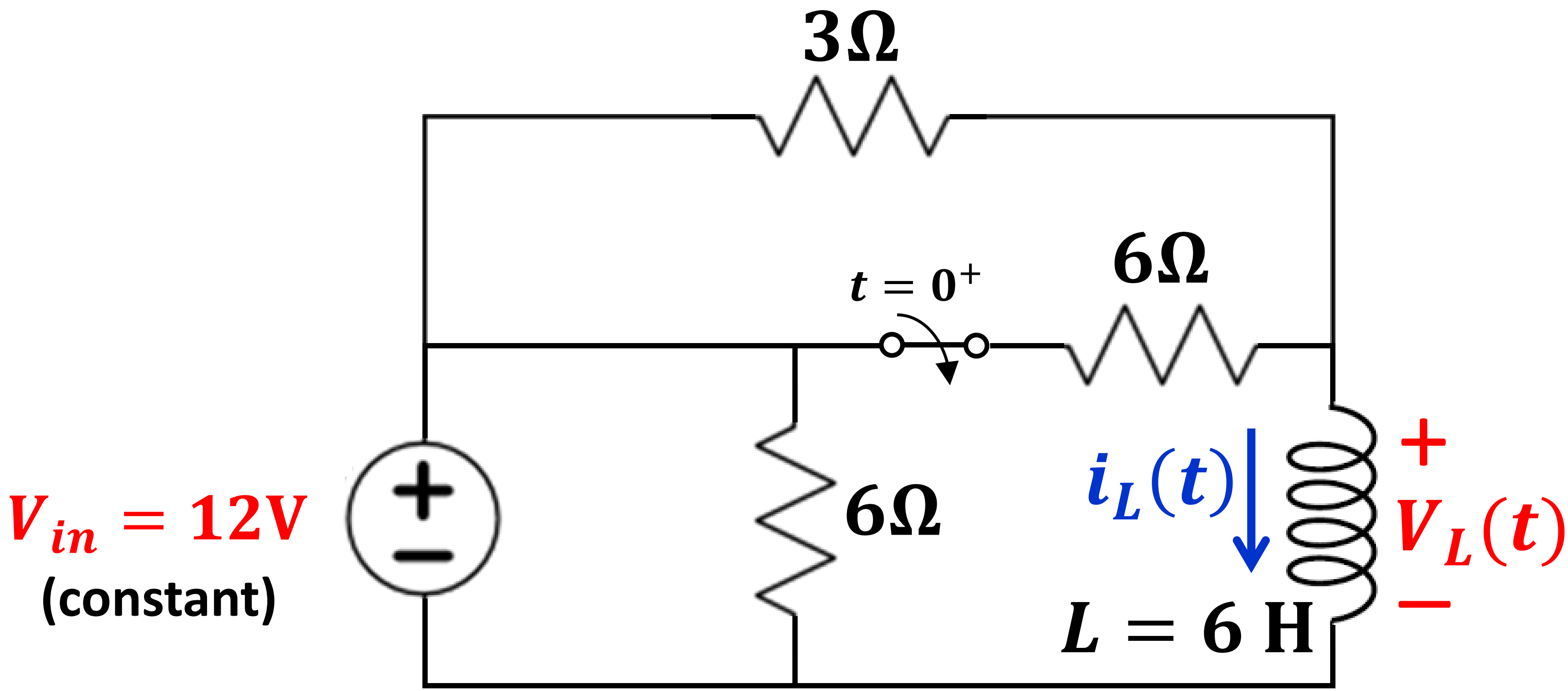


Step ② Find $i_L(0^+)$

$$i_L(0^+) = i_L(0^-) = K_1 + K_2 = 4\text{ A}$$



Can you tell what is $V_L(0^+)$?

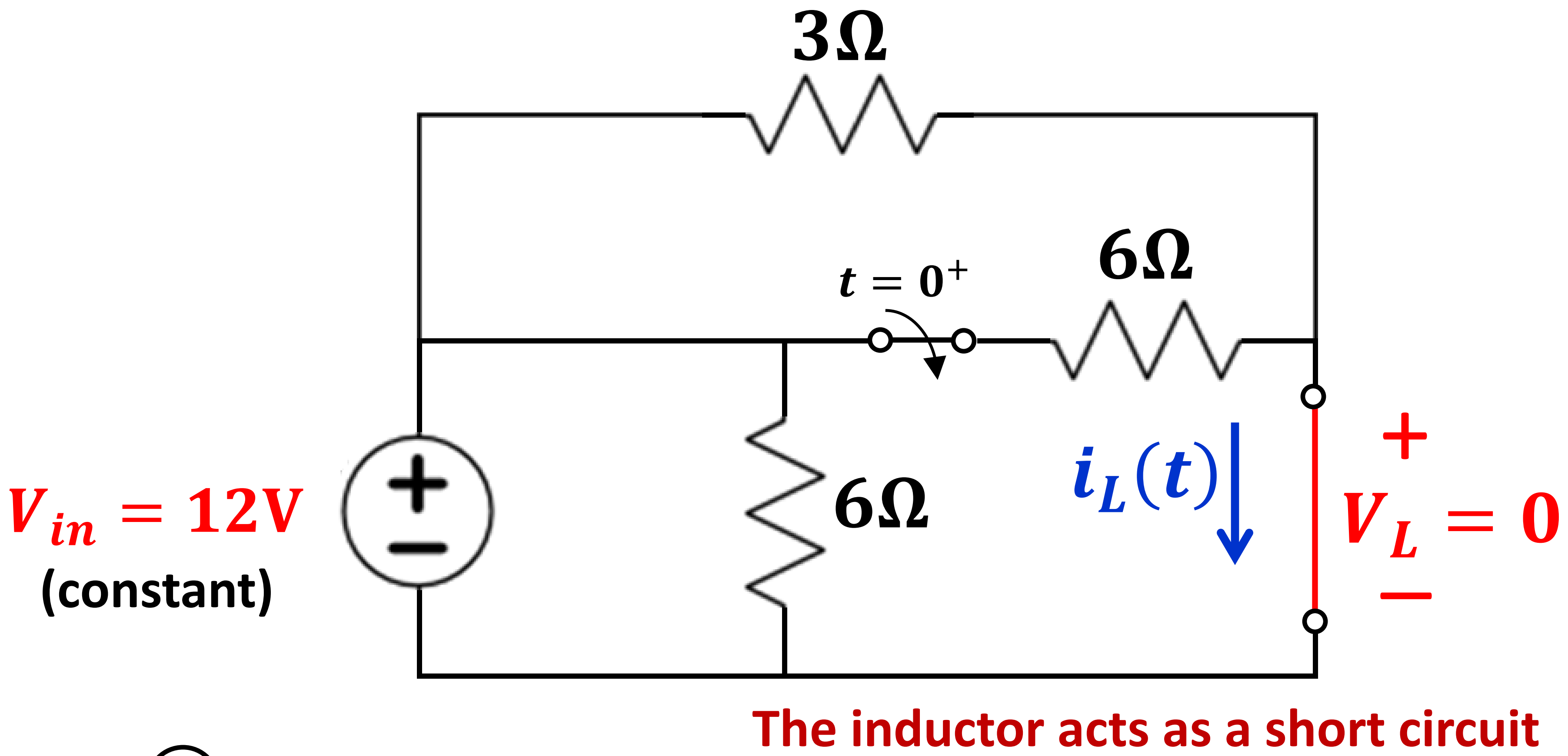


Can you tell what is $V_L(0^+)$? Write the KVL:

$$-V_{in} + i_L(0^+) \times (3\Omega//6\Omega) + V_L(0^+) = 0$$

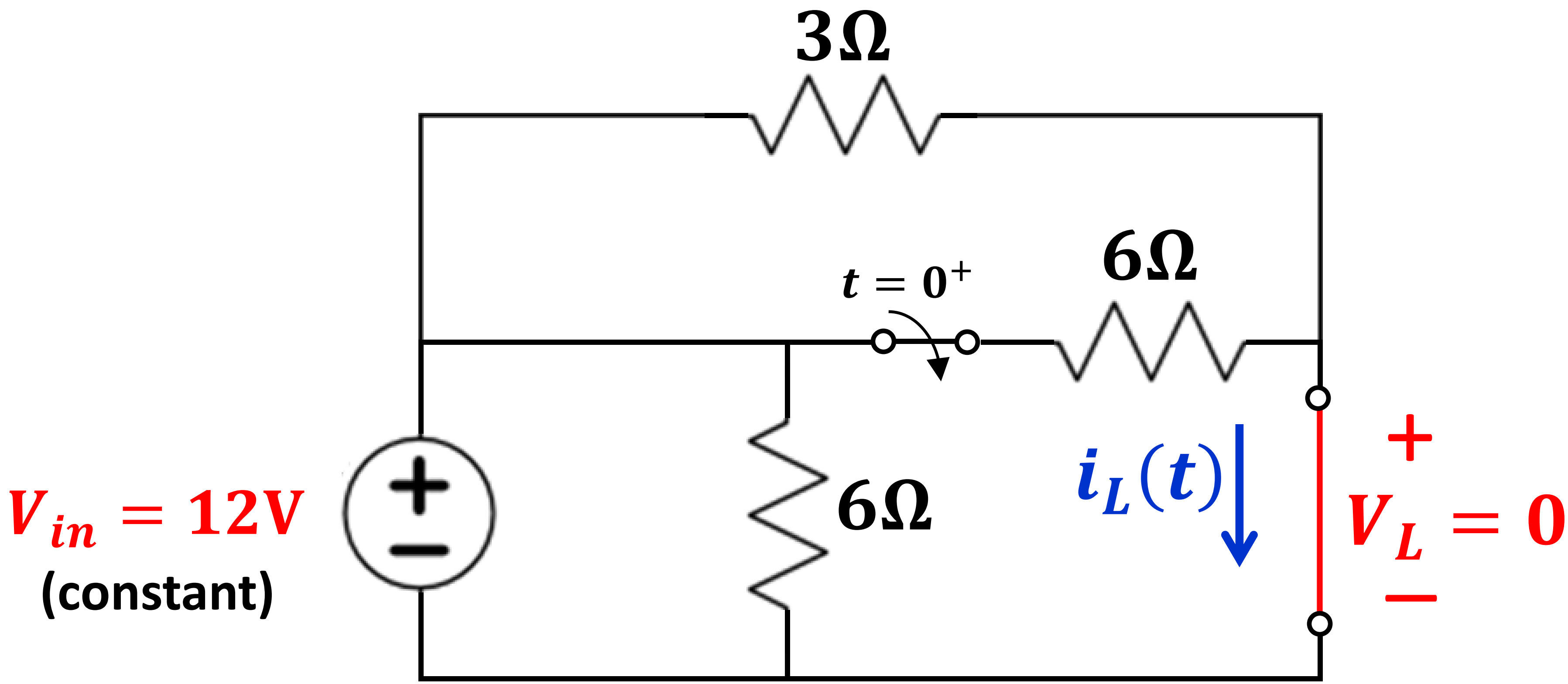
$$V_L(0^+) = V_{in} - i_L(0^+) \times (3\Omega//6\Omega)$$

$$V_L(0^+) = 12 - 4 \times 2 = 4\text{V}$$



Step ③ Find $i_L(\infty)$

$$i_L(\infty) = K_2 = V_{in}/(3\Omega//6\Omega) = 12/2 = 6\ \text{A}$$



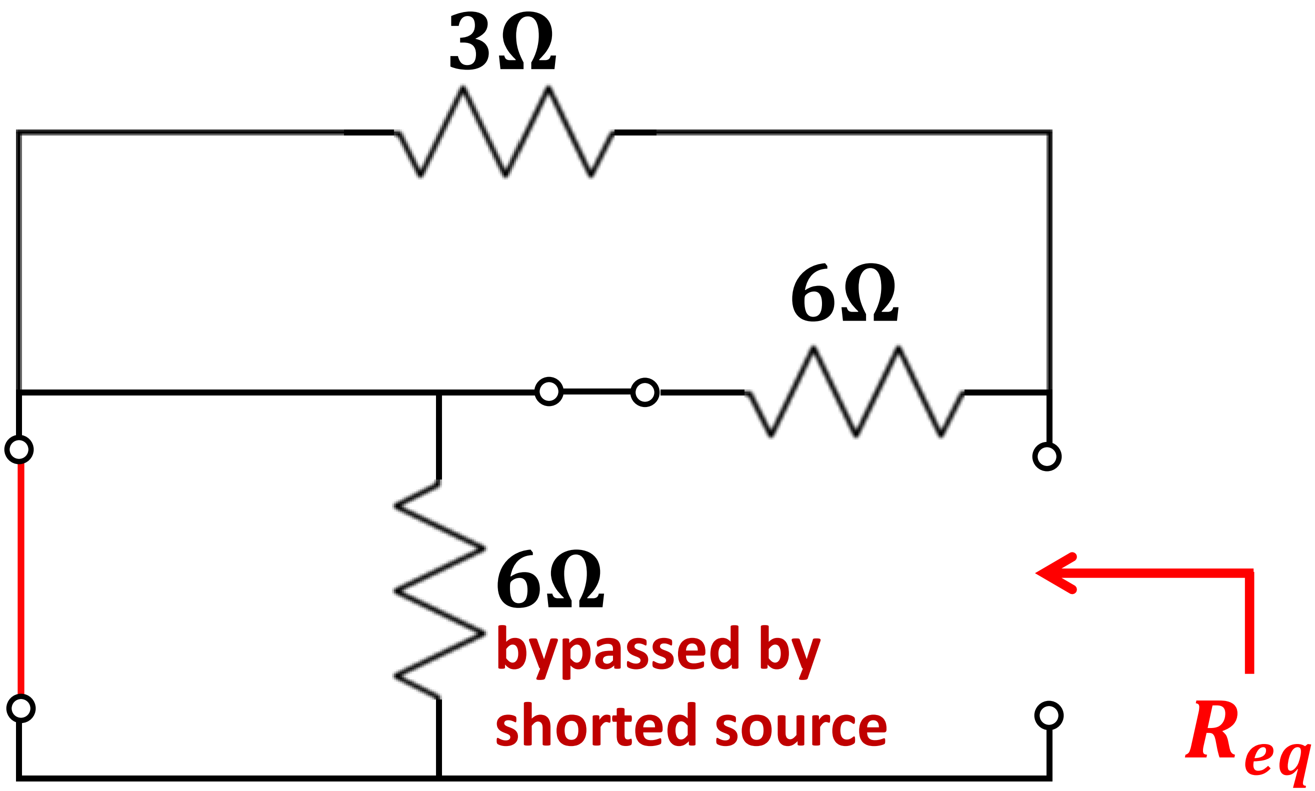
The inductor acts as a short circuit at $t \to \infty$

Step (3) Find $i_L(\infty)$

$$i_L(\infty) = K_2 = V_{in}/(3\Omega//6\Omega) = 12/2 = 6\text{ A}$$

$$i_L(0^+) = i_L(0^-) = K_1 + K_2 = 4\text{ A}$$

→ $K_2 = 6\text{ A}$ $K_1 = -2\text{ A}$



Step ④ Find α

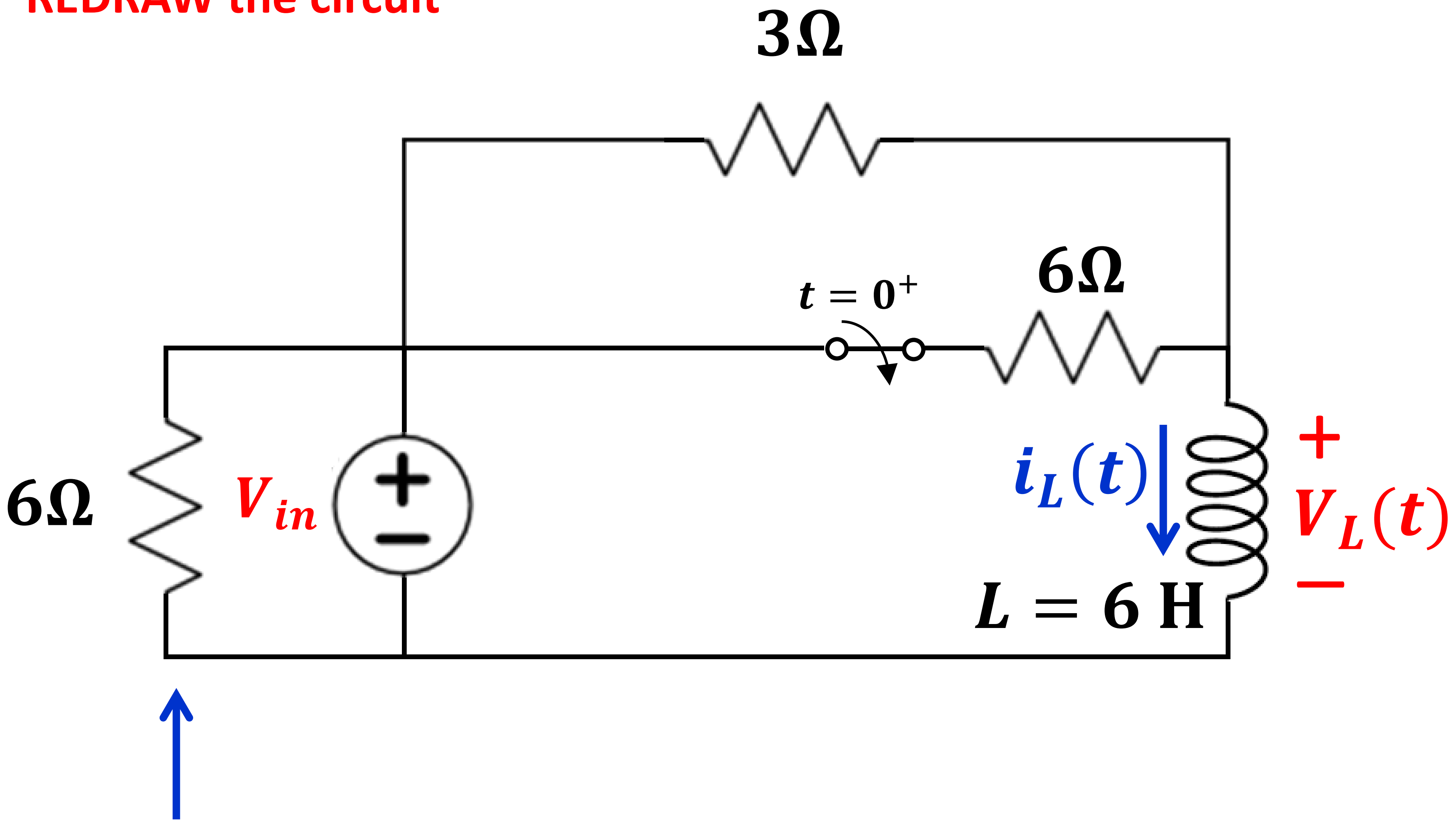
$$R_{eq} = 3\Omega // 6\Omega = 2\ \Omega$$

$$\alpha = \frac{R_{eq}}{L} = \frac{2}{6} = \frac{1}{3} s^{-1}$$

Step ⑤ Find $i_L(t)$

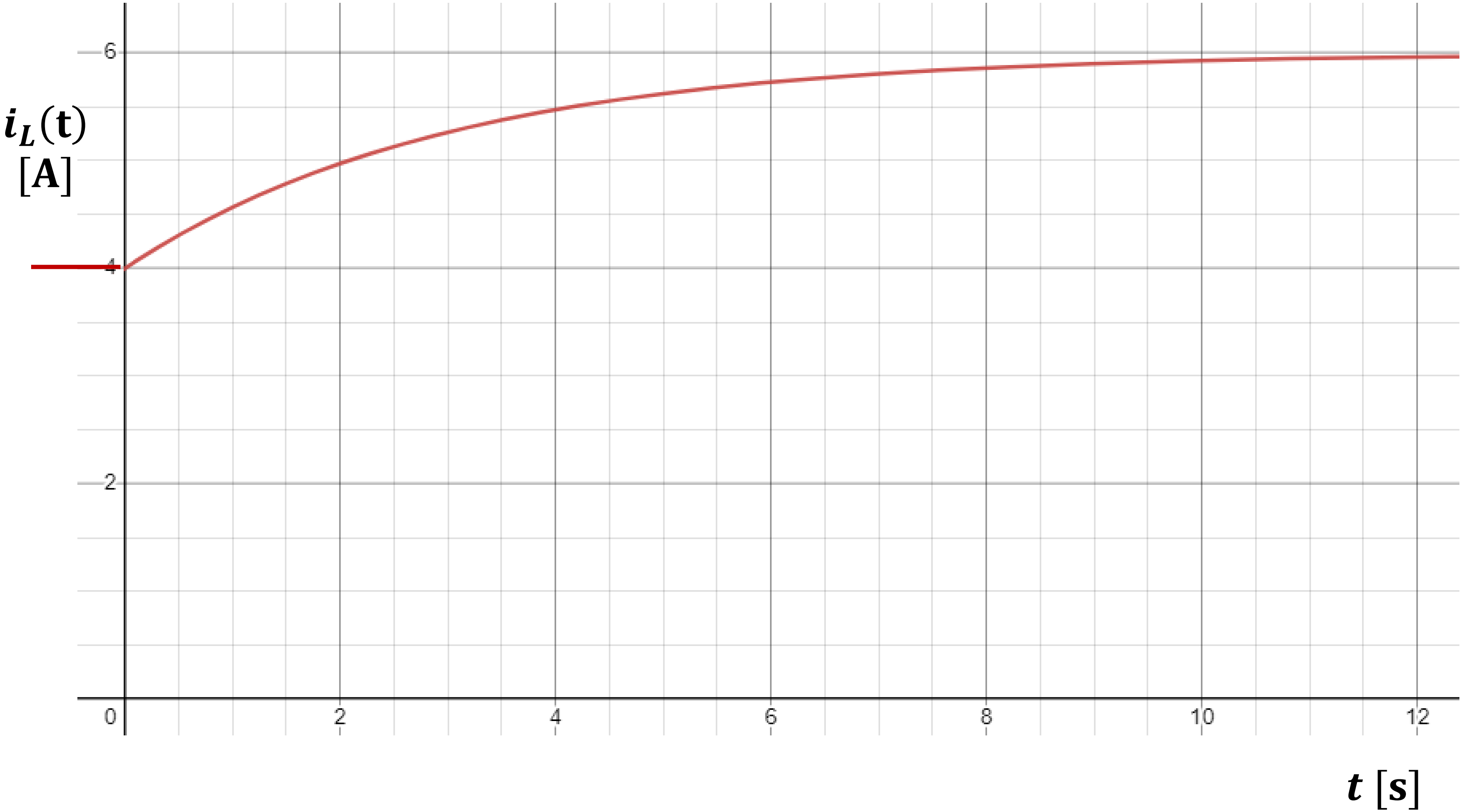
$$i_L(t) = -2e^{-t/3} + 6\ [\mathrm{A}]$$

REDRAW the circuit



NOTE: This resistor does not affect the rest of the circuit because it is in parallel with the voltage source.

$$i_L(t) = -2e^{-t/3} + 6 \text{ [A]}$$

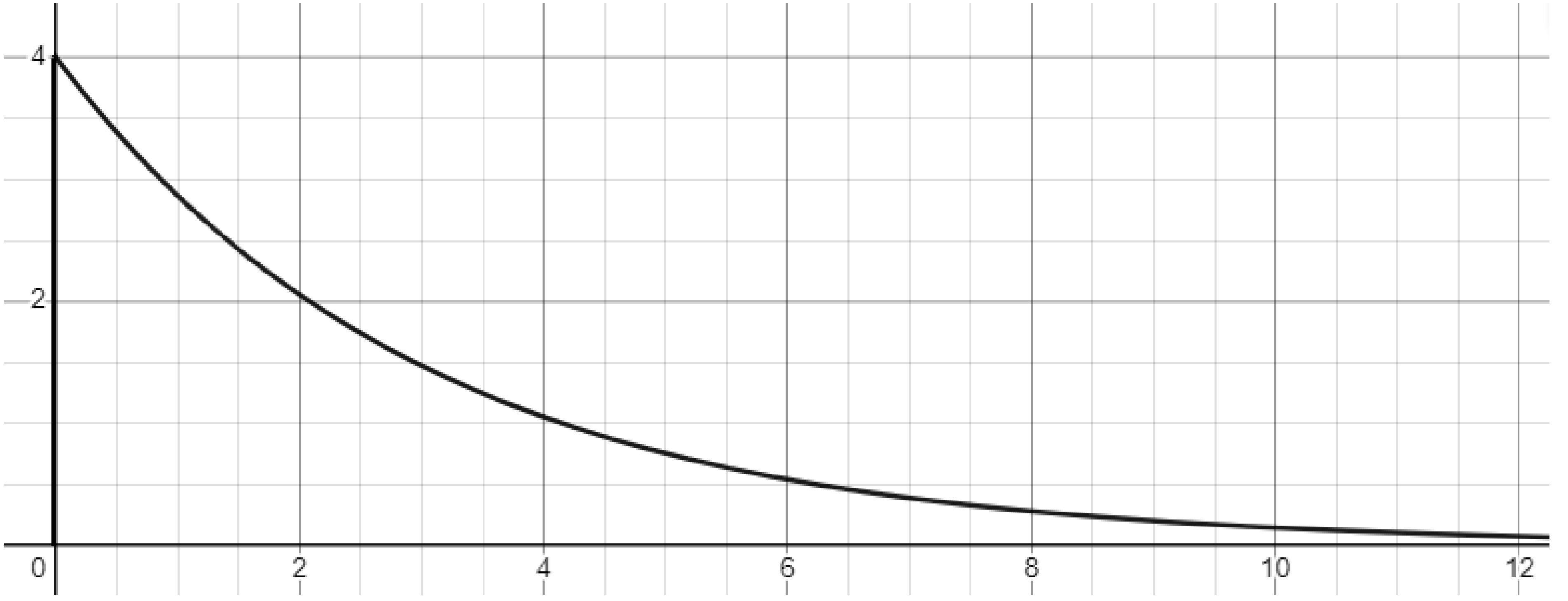


$$i_L(t) = -2e^{-t/3} + 6\ [\mathrm{A}]$$

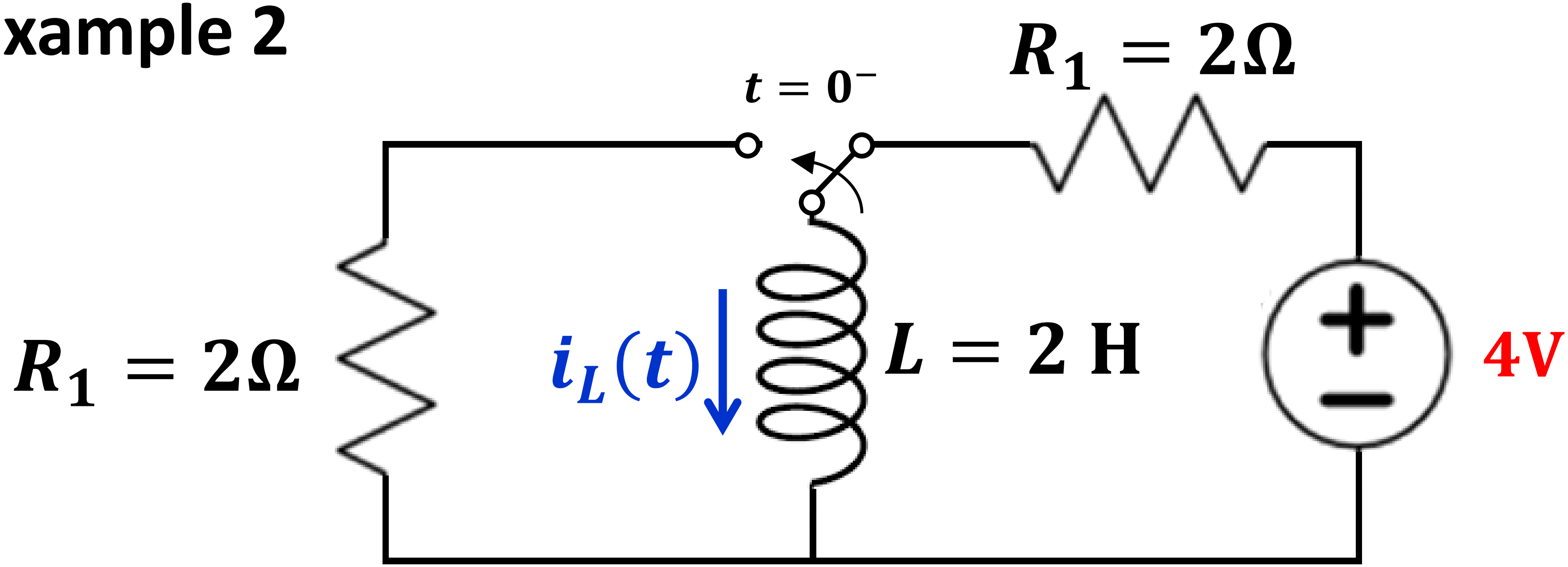
$$L = 6\ \mathrm{H}$$

Find the inductor voltage

$$V_L(t) = L\,\frac{d}{dt}i_L(t) = 6 \times \frac{2}{3}e^{-t/3}\ [\mathrm{V}]$$



Example 2

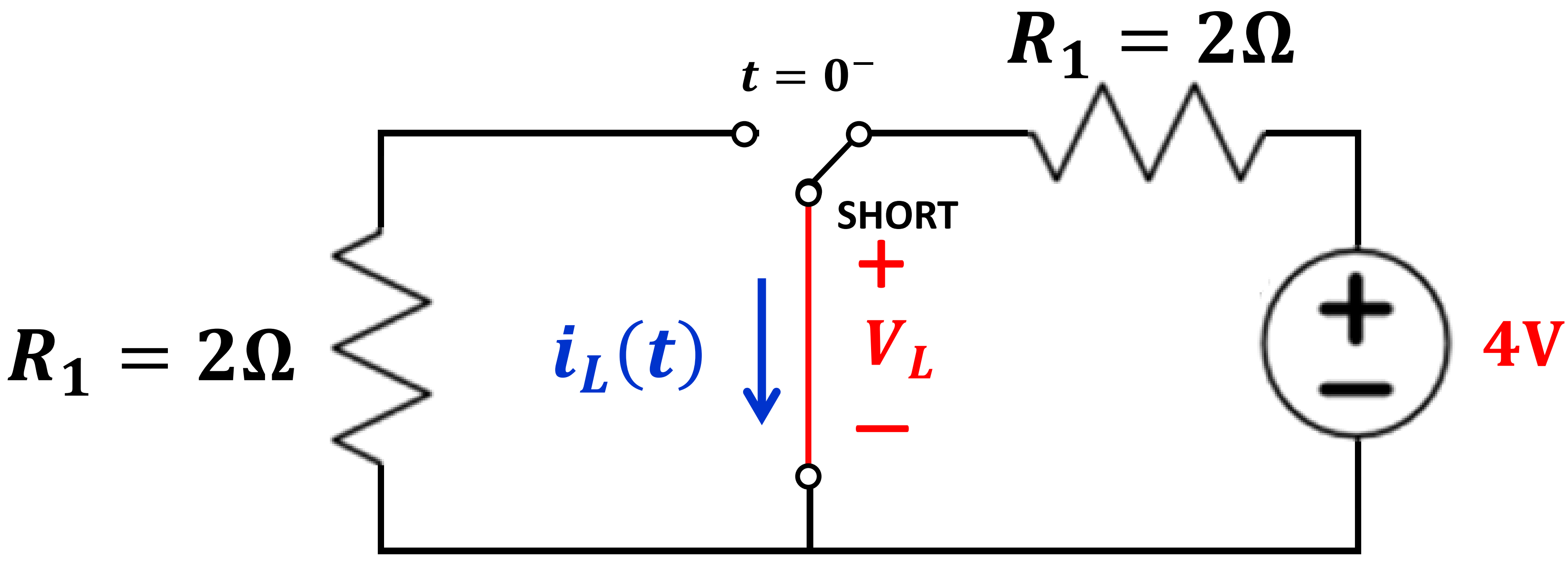


(constant DC source)

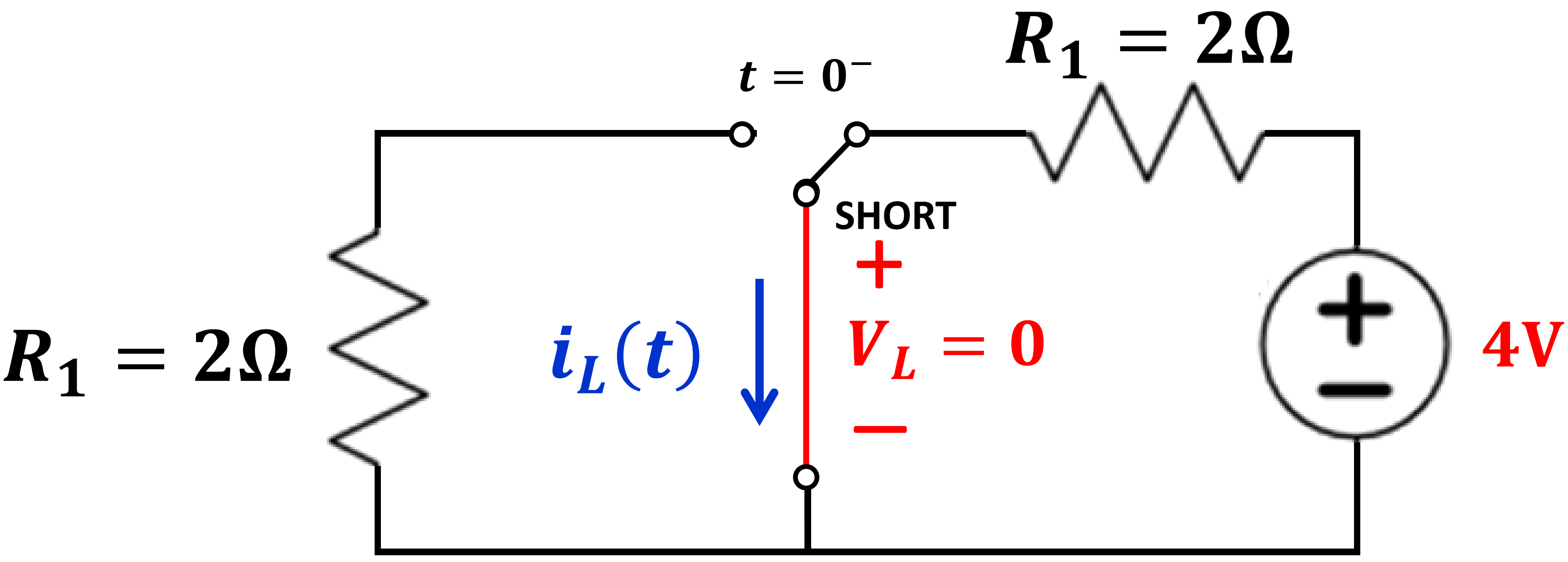
Find $i_L(t)$

Switch moves to the left position at $t = 0^+$

$$i_L(t) = K_1 e^{-\alpha t} + K_2 \quad [\mathrm{A}]$$



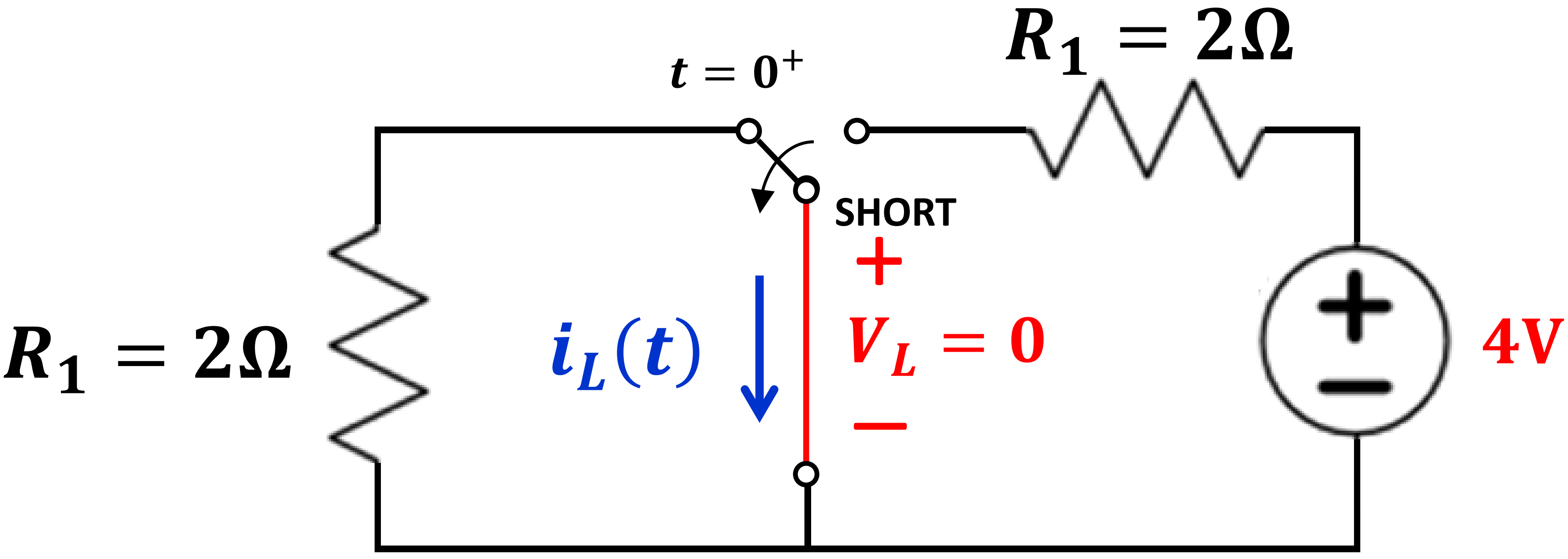
Step ① **Find $i_L(0^-)$ and $V_L(0^-)$ before the switch is closed**



Step ① Find $i_L(0^-)$ and $V_L(0^-)$ before the switch is closed

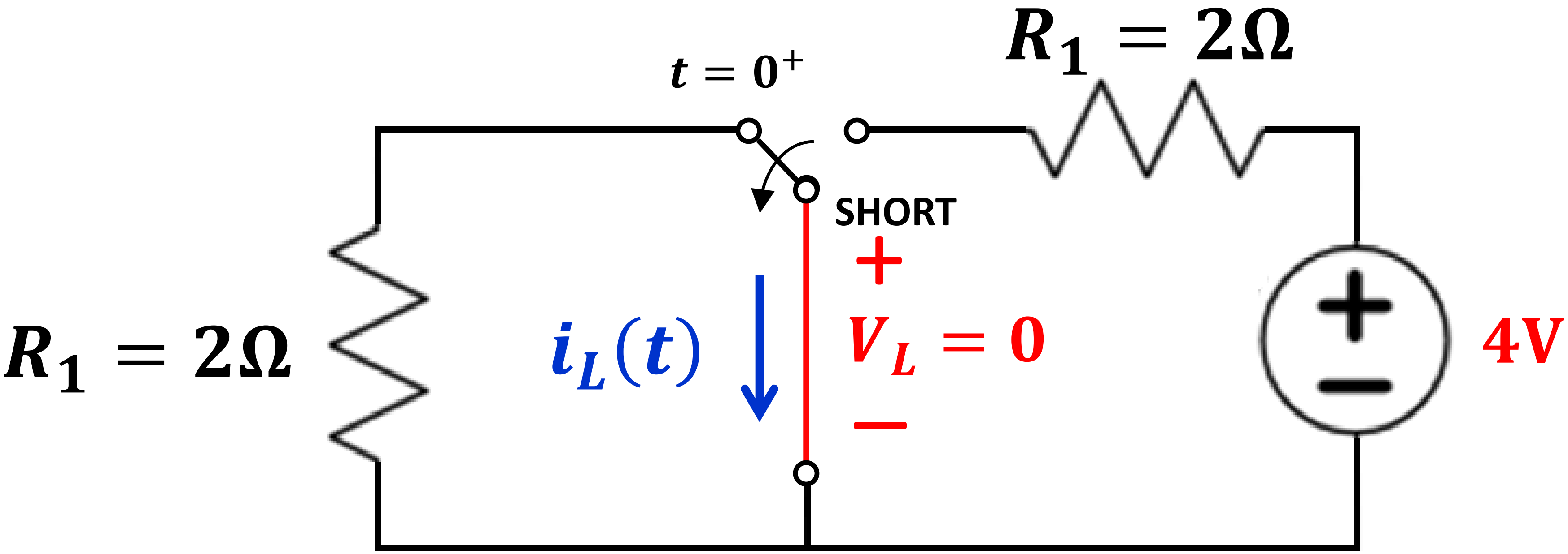
$$i_L(0^-) = \frac{4\text{V}}{2\Omega} = 2\ [\text{A}]$$

$$V_L(0^-) = 0\ \text{V}$$



Step ② Find $i_L(0^+)$

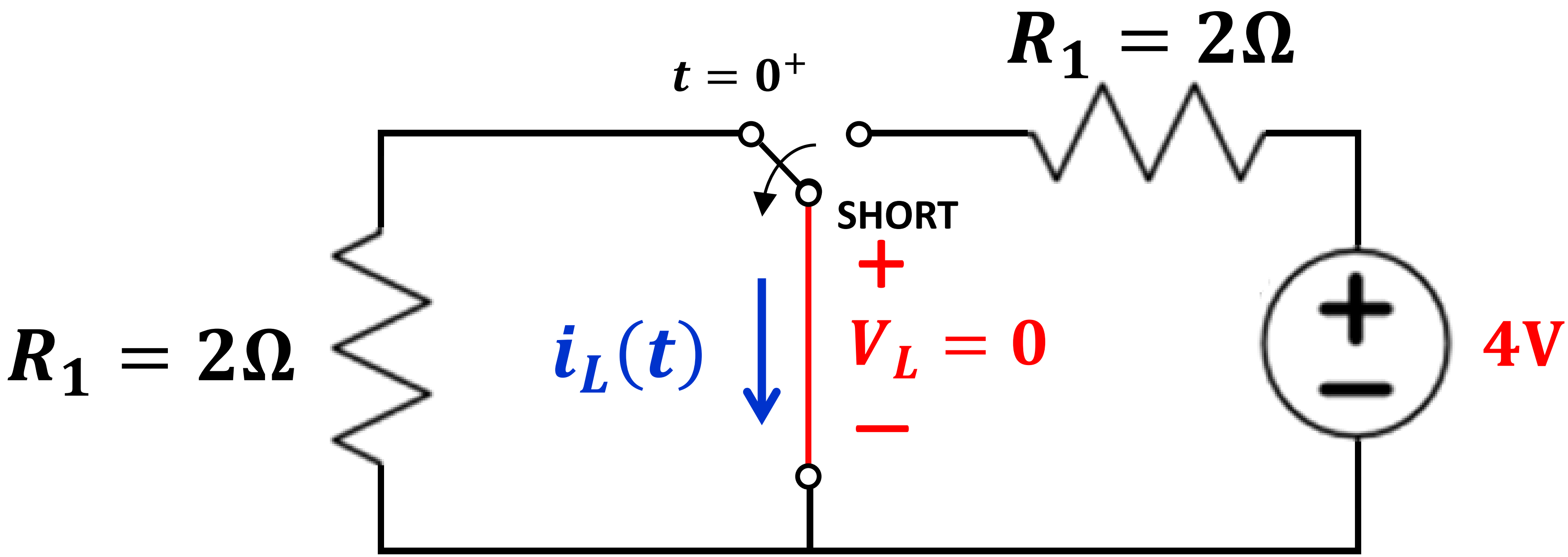
$$i_L(0^+) = i_L(0^-) = K_1 + K_2 = 2\ \text{A}$$



Step ② Find $i_L(0^+)$

$$i_L(0^+) = i_L(0^-) = K_1 + K_2 = 2\ \text{A}$$

What is $V_L(0^+)$?



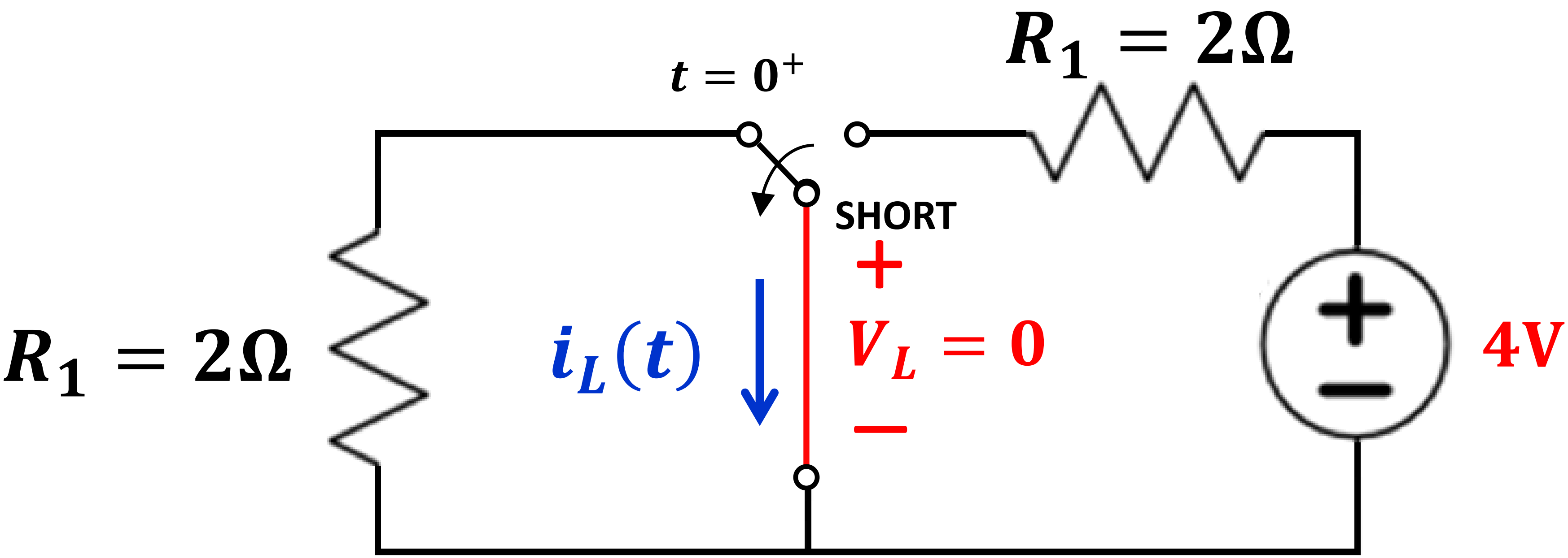
Step ② Find $i_L(0^+)$

$$i_L(0^+) = i_L(0^-) = K_1 + K_2 = 2\ \text{A}$$

What is $V_L(0^+)$? Write the KVL:

$$V_L(0^+) + R_1 \times i_L(0^+) = 0$$

$$V_L(0^+) = -R_1 \times i_L(0^+) = -2 \times 2 = -4\text{V}$$



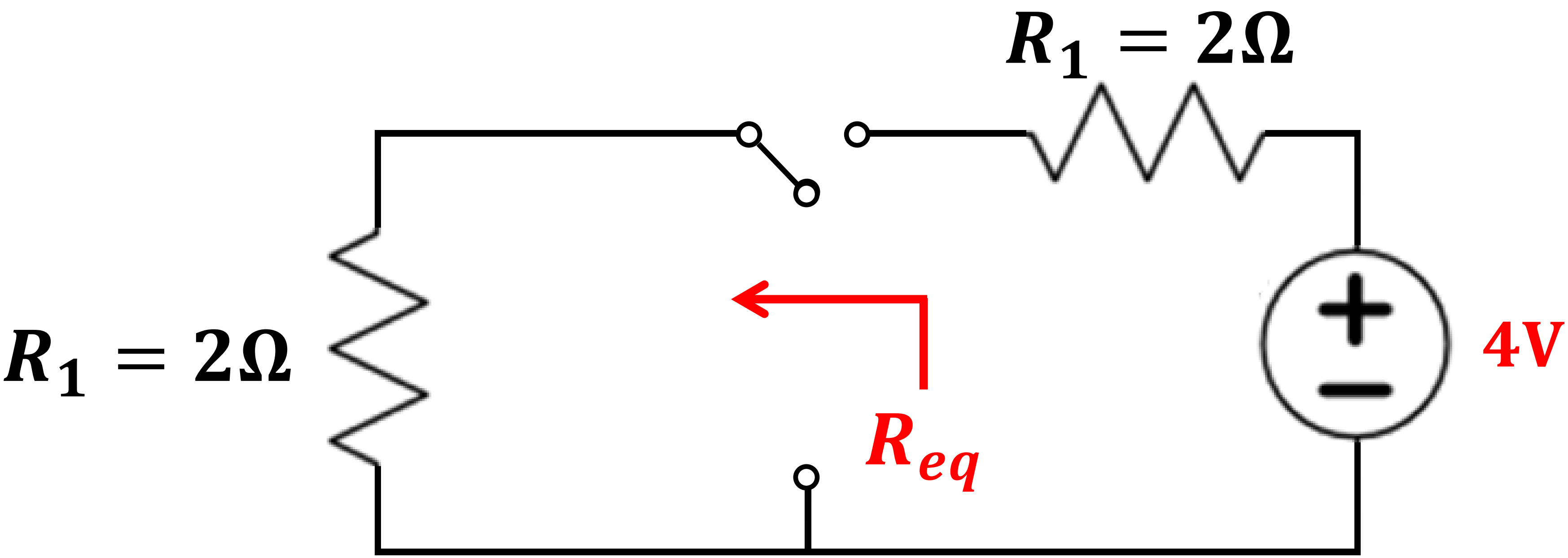
Step ② Find $i_L(0^+)$

$$i_L(0^+) = i_L(0^-) = K_1 + K_2 = 2\ \text{A}$$

Step ③ Find $i_L(\infty)$

$$i_L(\infty) = K_2 = 0\ \text{A}$$

→ $K_2 = 0\ \text{A}$ $K_1 = 2\ \text{A}$



Step ④ Find α

$$R_{eq} = 2\ \Omega$$

$$\alpha = \frac{R_{eq}}{L} = \frac{2\Omega}{2\text{H}} = 1\ \text{s}^{-1}$$

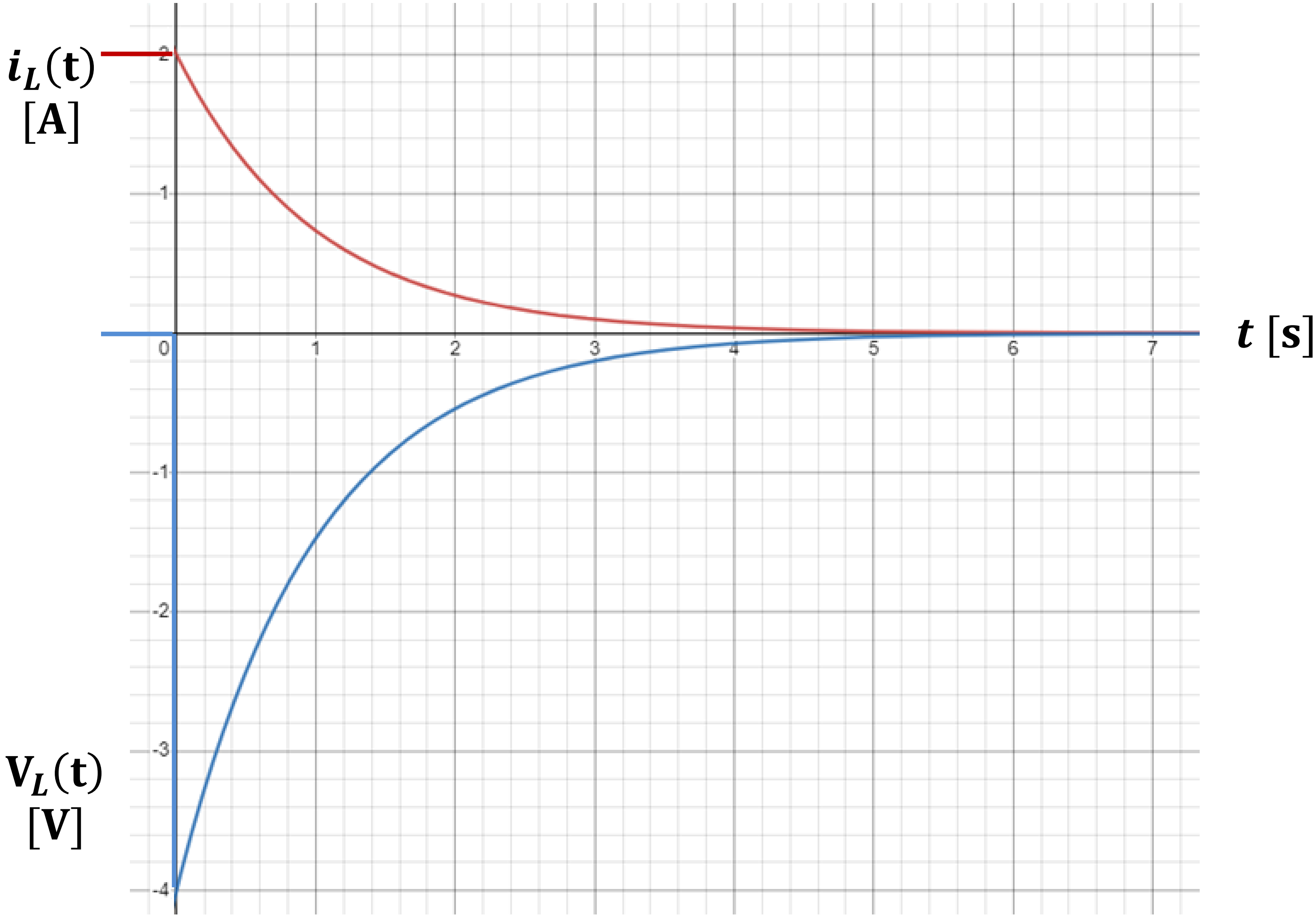
Step ⑤ Find $i_L(t)$

$$i_L(t) = K_1 e^{-\alpha t} + K_2 = 2e^{-t}\ [\text{A}]$$

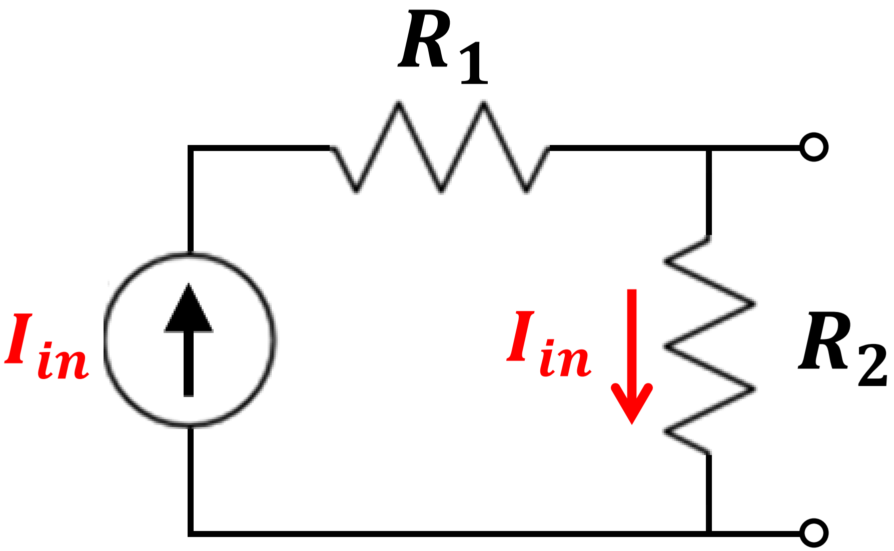
$$i_L(t) = K_1 e^{-\alpha t} + K_2 = 2e^{-t} \text{ [A]}$$

$$V_L(t) = L\, d(2e^{-t})/dt = -4e^{-t} \text{ [V]}$$

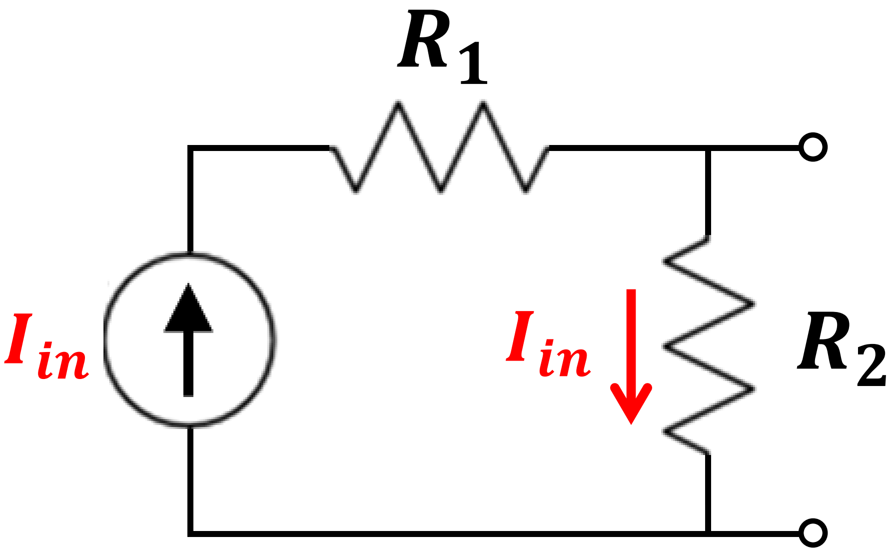
$L = 2$ H



Question: **What is the Thevenin equivalent of the circuit below?**



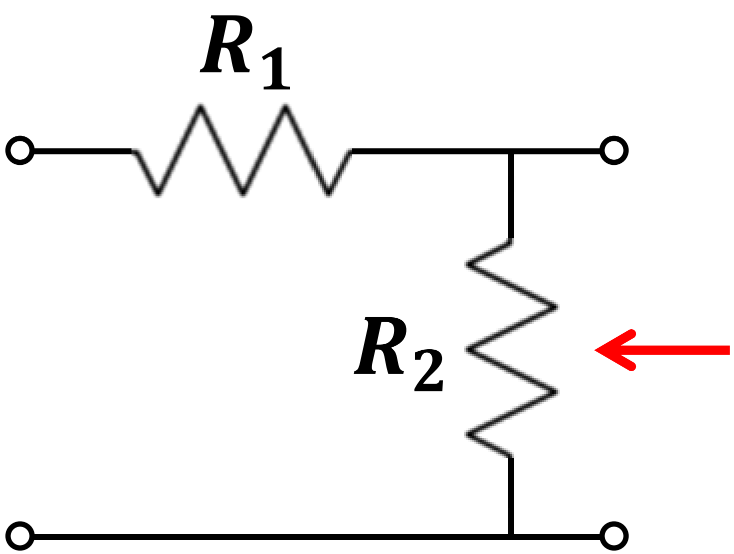
Question: What is the Thevenin equivalent of the circuit below?



Open circuit voltage

$$V_{TH} = R_2 \times I_{in}$$

R_1 does not play a role in determining the Thevenin circuit equivalent.



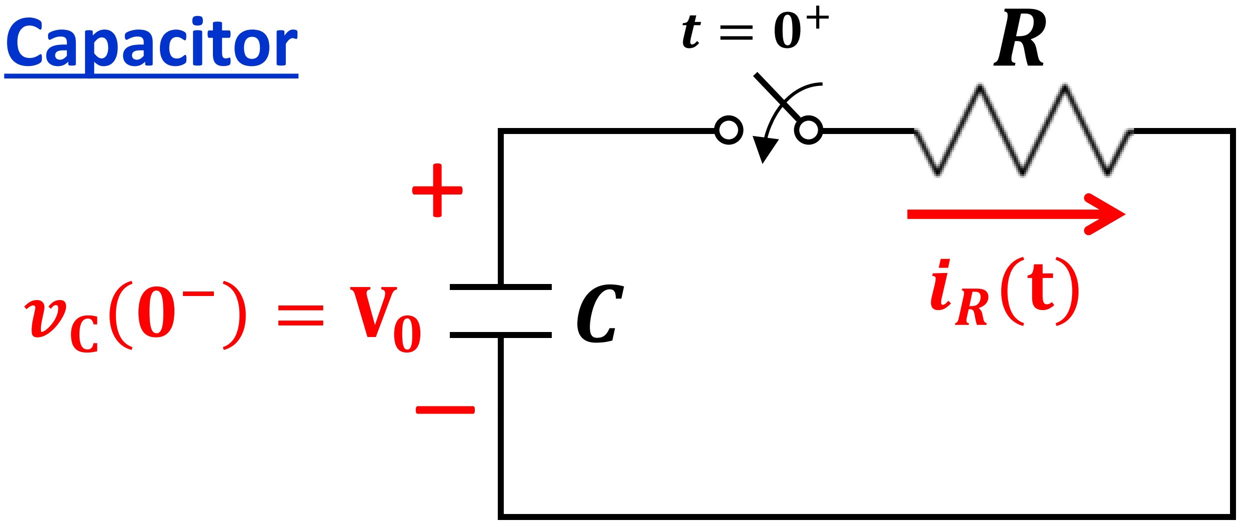
$$R_{TH} = R_2$$

This could be useful in HW6 Q3

Power Considerations

1. Energy stored by a capacitor
2. Energy stored by an inductor

Capacitor

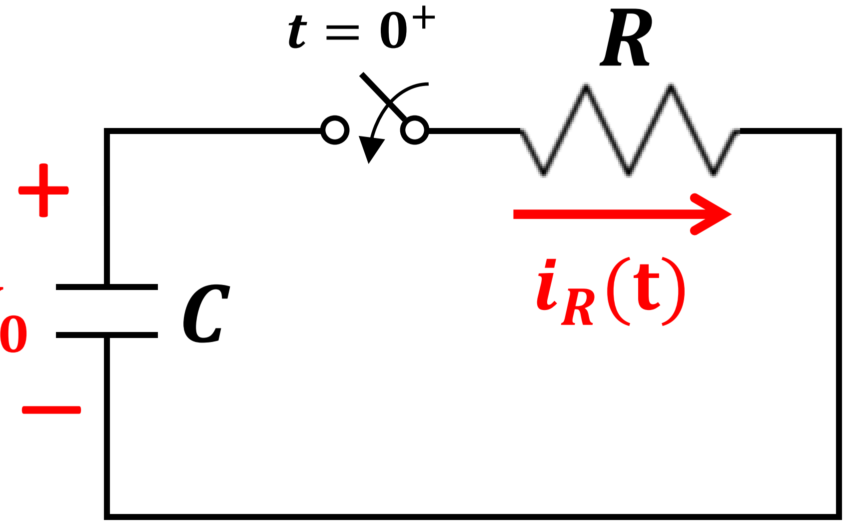


$$\alpha = \frac{1}{\tau} = \frac{1}{RC}$$

$$C = \frac{\tau}{R}$$

At $t \to \infty$ the resistor has dissipated all the energy stored in the capacitor

Capacitor



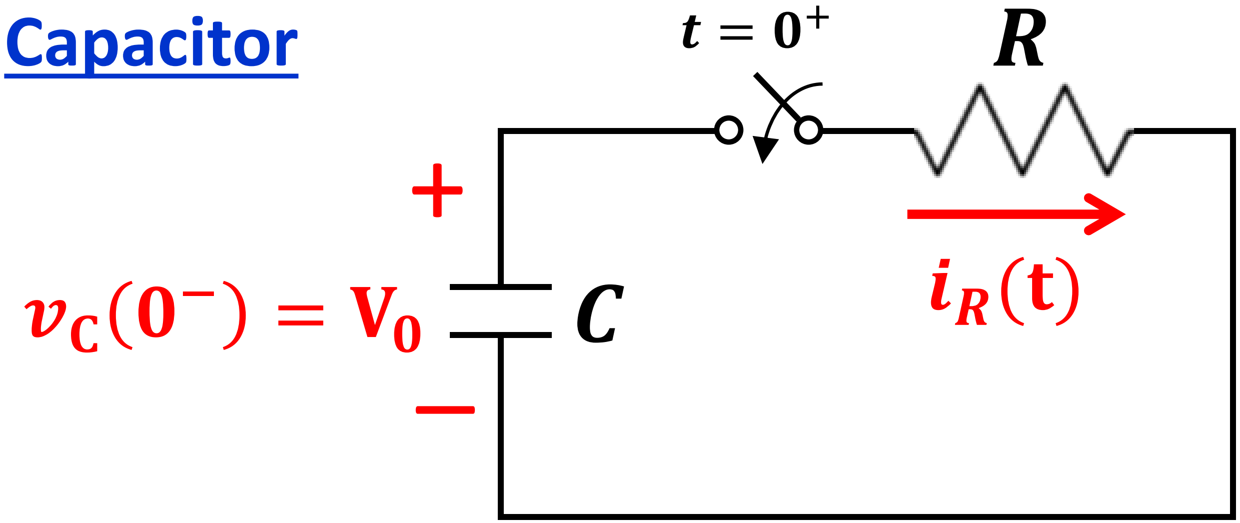
$$\alpha = \frac{1}{\tau} = \frac{1}{RC}$$

$$C = \frac{\tau}{R}$$

At $t \to \infty$ the resistor has dissipated all the energy stored in the capacitor

Power $$p_R(t) = v_C(t) \cdot i_R(t) = \frac{v_c^2(t)}{R} = \frac{V_0^2 e^{-2\alpha t}}{R}$$

Capacitor



$$\alpha = \frac{1}{\tau} = \frac{1}{RC}$$

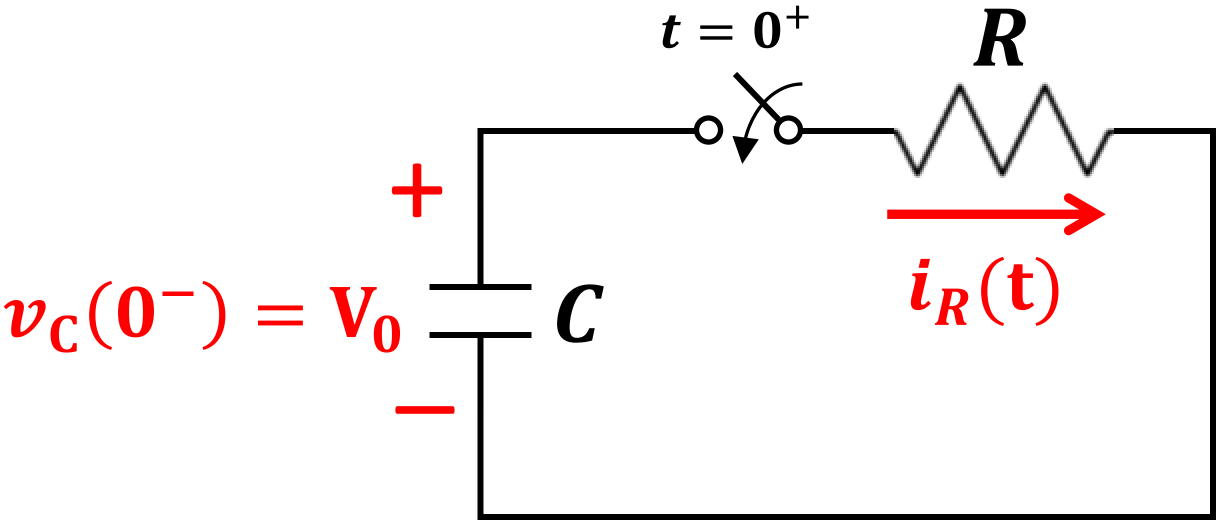
$$C = \frac{\tau}{R}$$

At $t \to \infty$ the resistor has dissipated all the energy stored in the capacitor

Power $$p_R(t) = v_C(t) \cdot i_R(t) = \frac{v_c^2(t)}{R} = \frac{V_0^2 e^{-2\alpha t}}{R}$$

Energy absorbed $$w_R(t) = \int_0^t p_R(t')\,dt' = \int_0^t \frac{V_0^2 e^{-2\alpha t'}}{R}\,dt'$$

$$= -\frac{\tau V_0^2}{2R} e^{-2\alpha t}\Big|_0^t = \frac{1}{2} C\, V_0^2\left[1 - e^{-2\alpha t}\right]$$



$$\alpha = \frac{1}{\tau} = \frac{1}{RC}$$

$$C = \frac{\tau}{R}$$

Energy absorbed

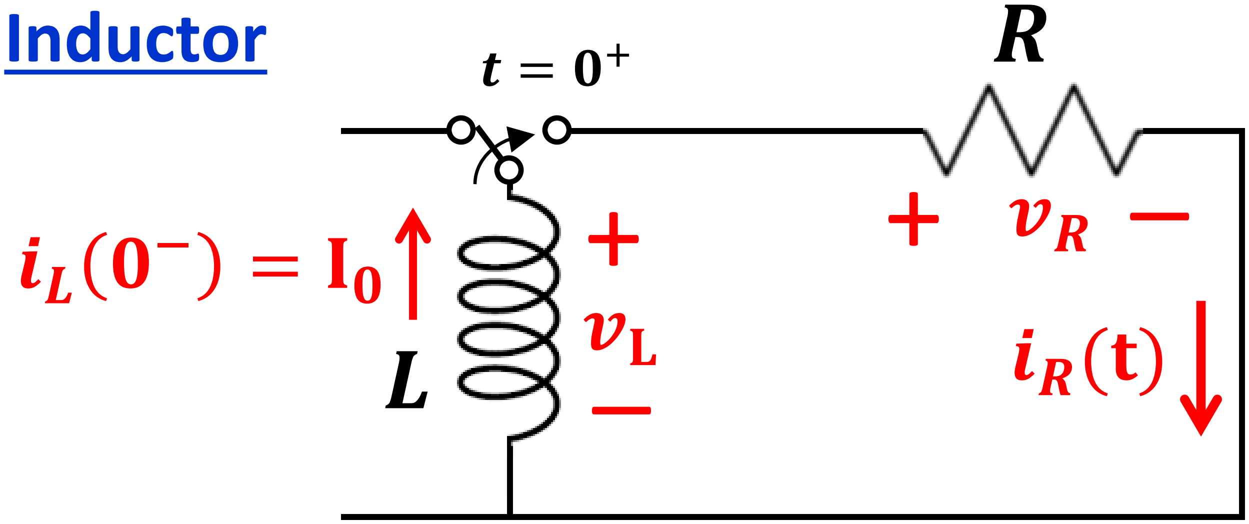
$$w_R(t) = \frac{1}{2} C\, V_0^2 \left[1 - e^{-2\alpha t}\right]$$

$t \to \infty$

$$w_R(t \to \infty) = \frac{1}{2} C\, V_0^2$$

Energy dissipated by R = Energy stored by C

Inductor

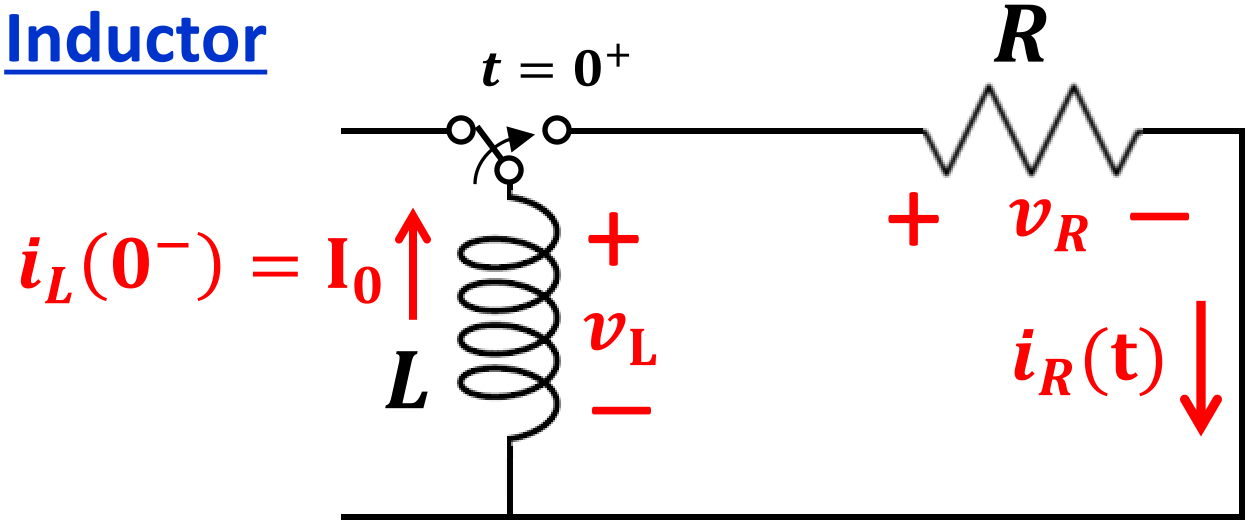


$$\alpha = \frac{1}{\tau} = \frac{R}{L}$$

$$L = \tau R$$

At $t \to \infty$ the resistor has dissipated all the energy stored in the inductor

Inductor



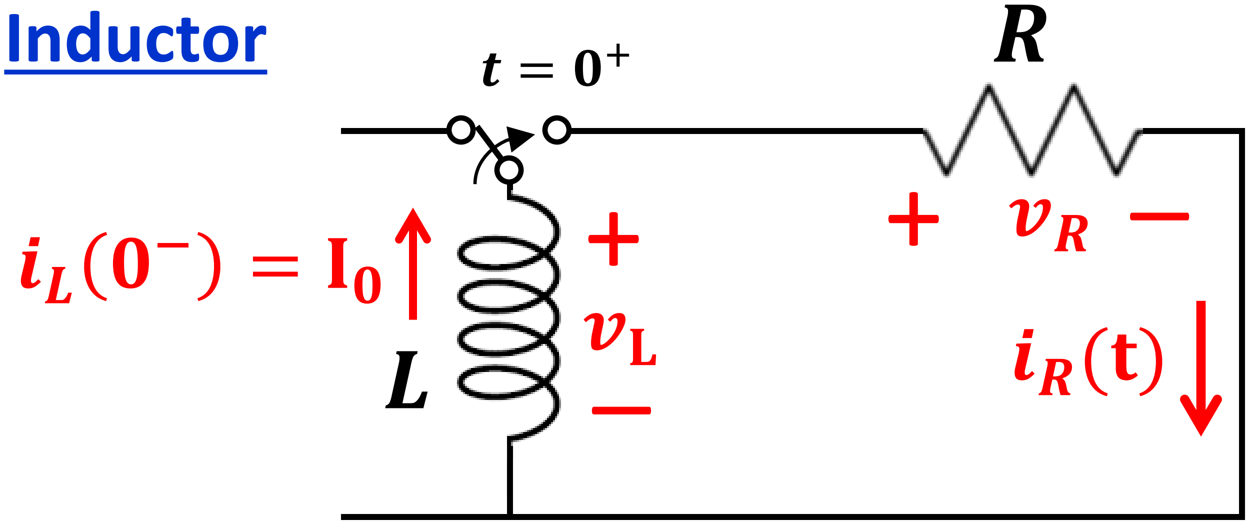
$$\alpha = \frac{1}{\tau} = \frac{R}{L}$$

$$L = \tau R$$

At $t \to \infty$ the resistor has dissipated all the energy stored in the inductor

Power $$p_R(t) = v_R(t) \cdot i_R(t) = i_L^2(t) R = R I_0^2 e^{-2\alpha t}$$

Inductor



$$\alpha = \frac{1}{\tau} = \frac{R}{L}$$

$$L = \tau R$$

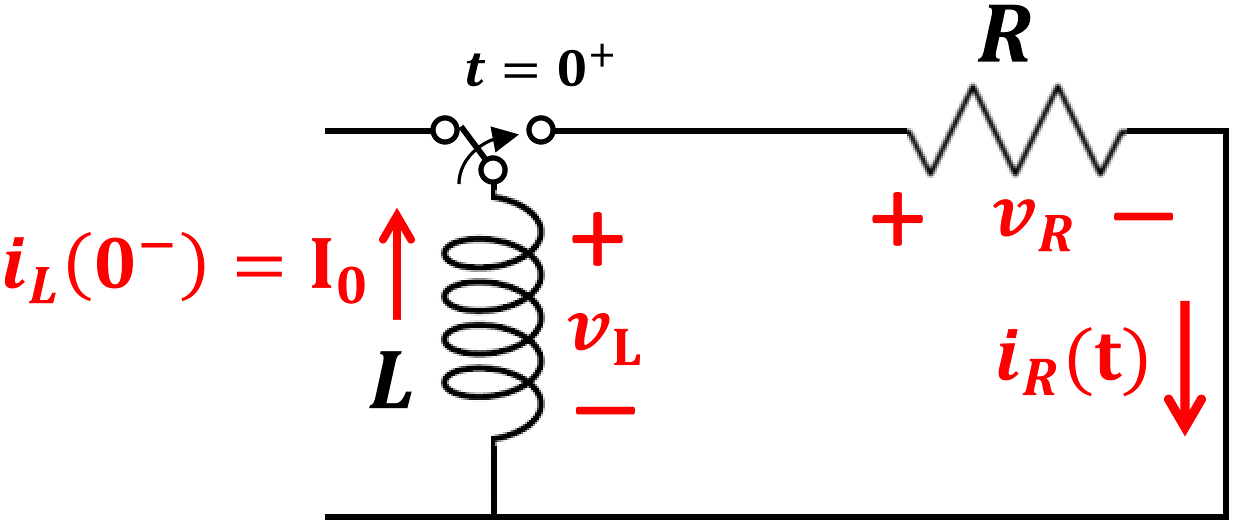
At $t \to \infty$ the resistor has dissipated all the energy stored in the inductor

Power $$p_R(t) = v_R(t) \cdot i_R(t) = i_L^2(t) R = R\mathrm{I}_0^2 e^{-2\alpha t}$$

Energy absorbed

$$w_R(t) = \int_0^t p_R(t')\,dt' = \int_0^t R\mathrm{I}_0^2 e^{-2\alpha t'}\,dt'$$

$$= -\frac{\tau R \mathrm{I}_0^2}{2} e^{-2\alpha t}\Big|_0^t = \frac{1}{2} L\,\mathrm{I}_0^2\left[1 - e^{-2\alpha t}\right]$$



$$\alpha = \frac{1}{\tau} = \frac{R}{L}$$

$$L = \tau R$$

Energy absorbed

$$w_R(t) = \frac{1}{2} L\, I_0^2 \left[1 - e^{-2\alpha t}\right]$$

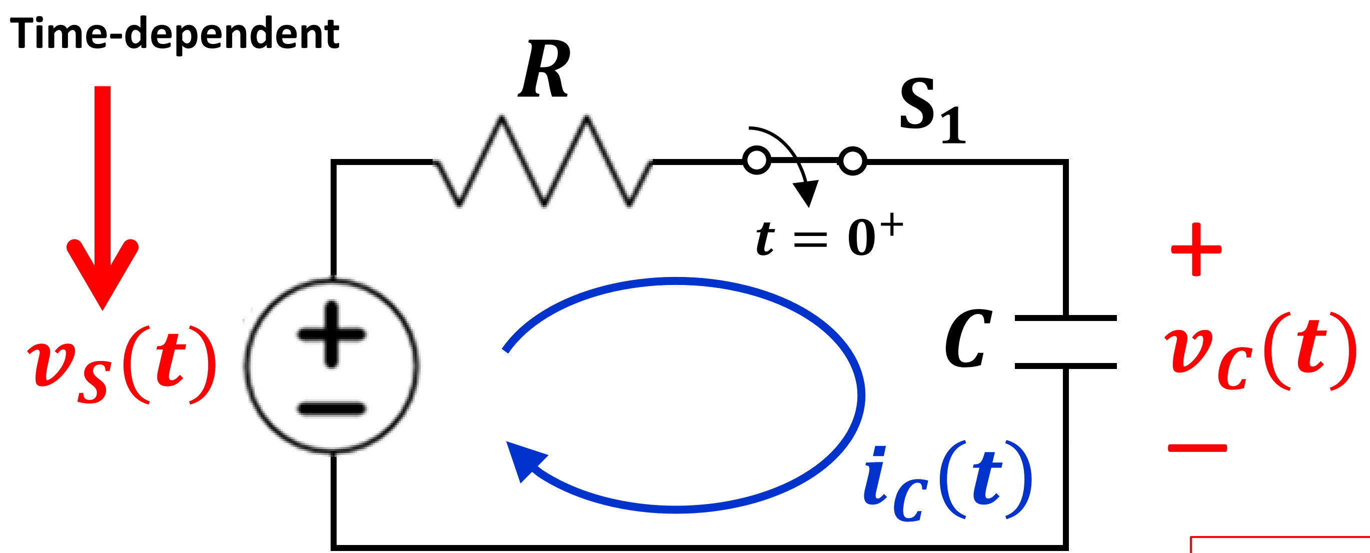
$t \to \infty$

$$w_R(t \to \infty) = \frac{1}{2} L\, I_0^2$$

Energy dissipated by R = Energy stored by L

RC and RL circuits with time-dependent input

RC circuit



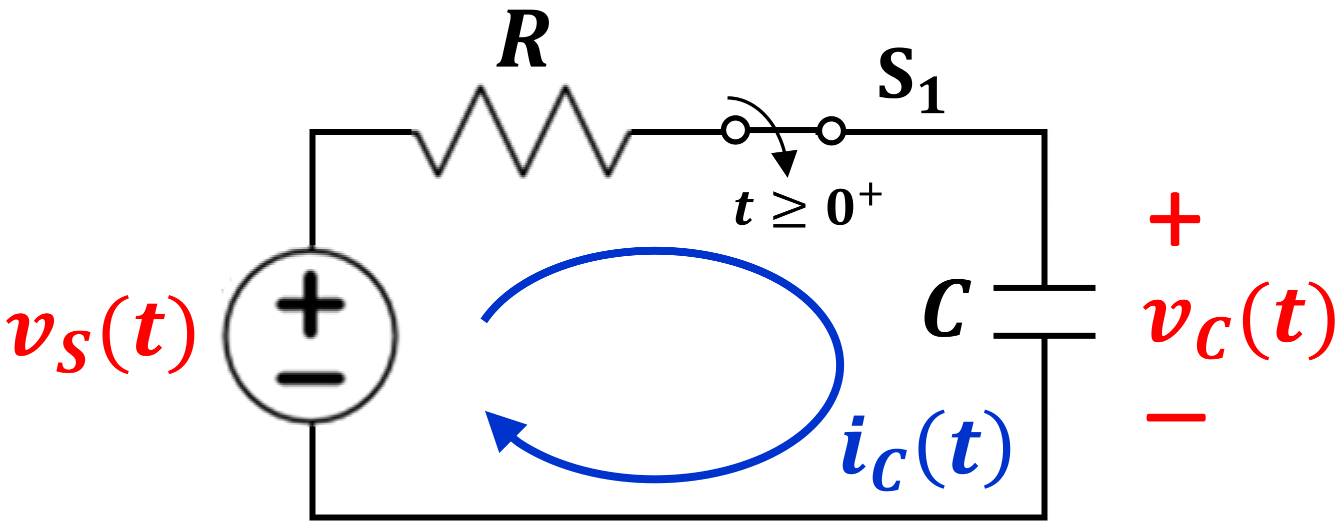
$$i_C(t) = C\frac{dv_C(t)}{dt}$$

KVL for the RC circuit

$$-v_S(t) + Ri_C(t) + v_C(t) = 0$$

$$RC\frac{d}{dt}v_C(t) + v_C(t) = v_S(t)$$

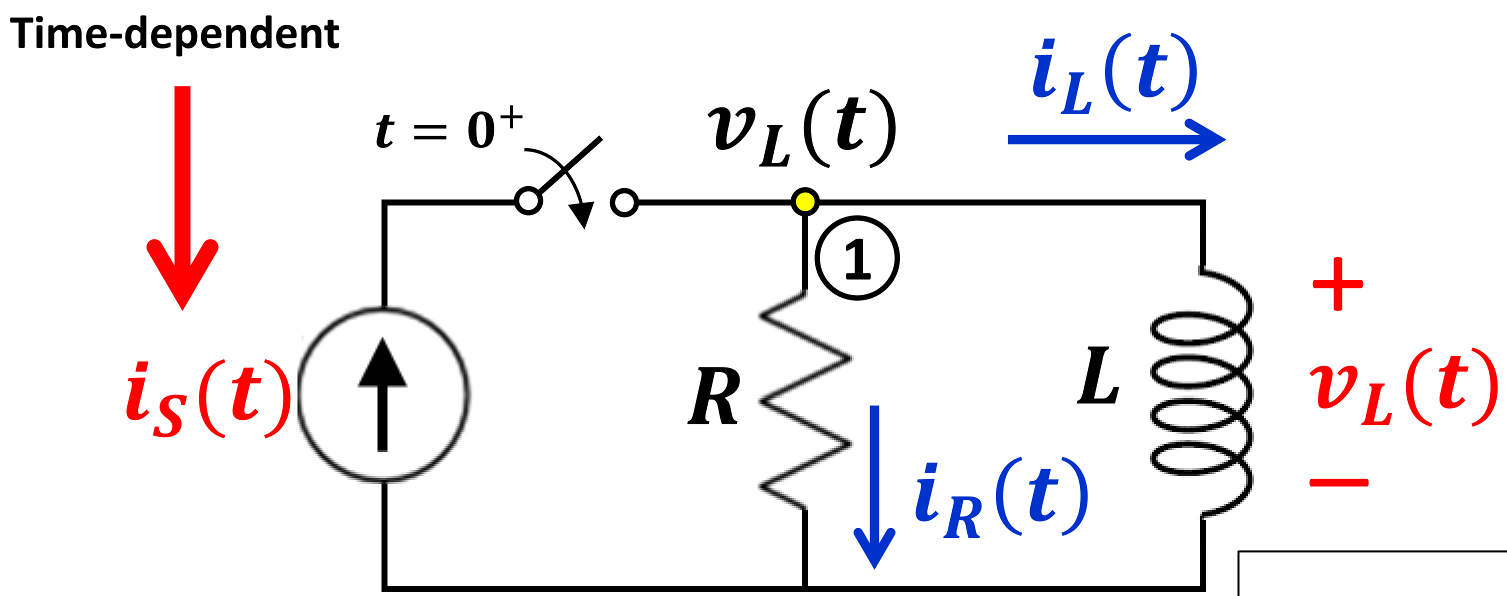
RC circuit



The RC circuit is described by the differential equation

$$\frac{d}{dt}v_C(t) + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_S(t)$$

RL circuit



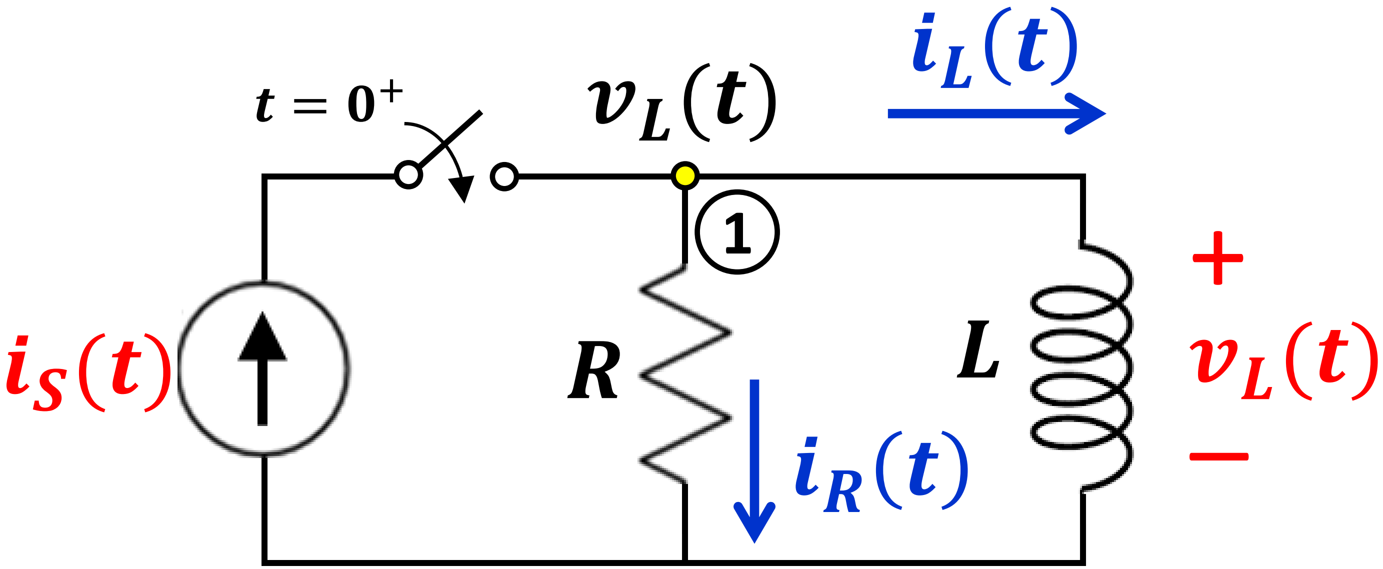
$$v_L(t) = L\frac{d}{dt}i_L(t)$$

KVL for the RL loop

$$v_L(t) - R[i_S(t) - i_L(t)] = 0$$

$$L\frac{d}{dt}i_L(t) + Ri_L(t) = Ri_S(t)$$

RL circuit



The RL circuit is described by the differential equation

$$\frac{d}{dt} i_L(t) + \frac{R}{L} i_L(t) = \frac{R}{L} i_S(t)$$

The transient behavior for RC and RL circuits is described by the first order ordinary differential equation

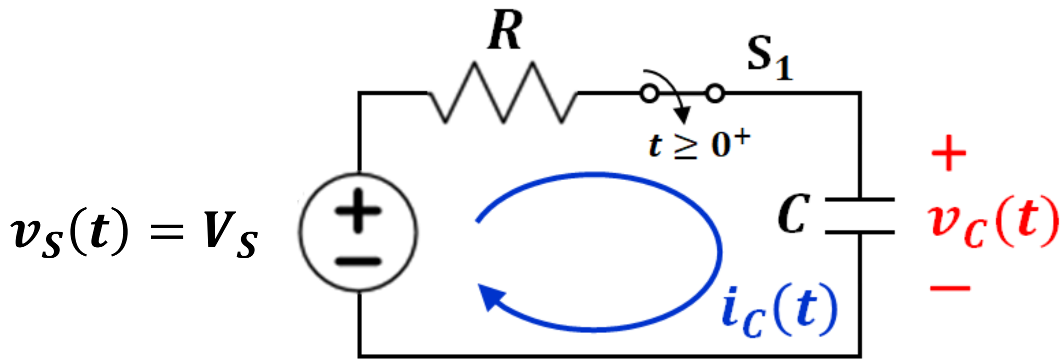
$$\frac{d}{dt}y(t) + a\,y(t) = b\,\underbrace{f(t)}$$

Forcing term

A solution satisfies the initial condition

$$y(0^-) = y(0^+)$$

CASE 1 – RC circuit with constant $v_S(t) = V_S$



$$\frac{d}{dt}V_C(t) + \frac{1}{RC}V_C(t) = \frac{1}{RC}V_S \qquad \boxed{1}$$

Initial condition

$$V_C(0^-) = V_C(0^+)$$

The solution can be written in the form

$$V_C(t) = u_h(t) + \underbrace{K_2}_{\text{constant}}$$

In our case, we have:

$$V_C(t) = V_h(t) + V_S$$

↓

$$\frac{d}{dt}[V_h + V_S] + \frac{1}{RC}[V_h + \cancel{V_S}] = \frac{1}{RC}\cancel{V_S}$$

↓

$$\frac{d}{dt}V_h(t) + \frac{1}{RC}V_h(t) = 0 \qquad \boxed{2}$$

Equation [2] is homogeneous and can be integrated as

$$V_h(t) = K_1\, e^{-t/RC}$$

Complete solution

$$V_c(t) = V_h(t) + K_2 = K_1\, e^{-t/RC} + K_2$$

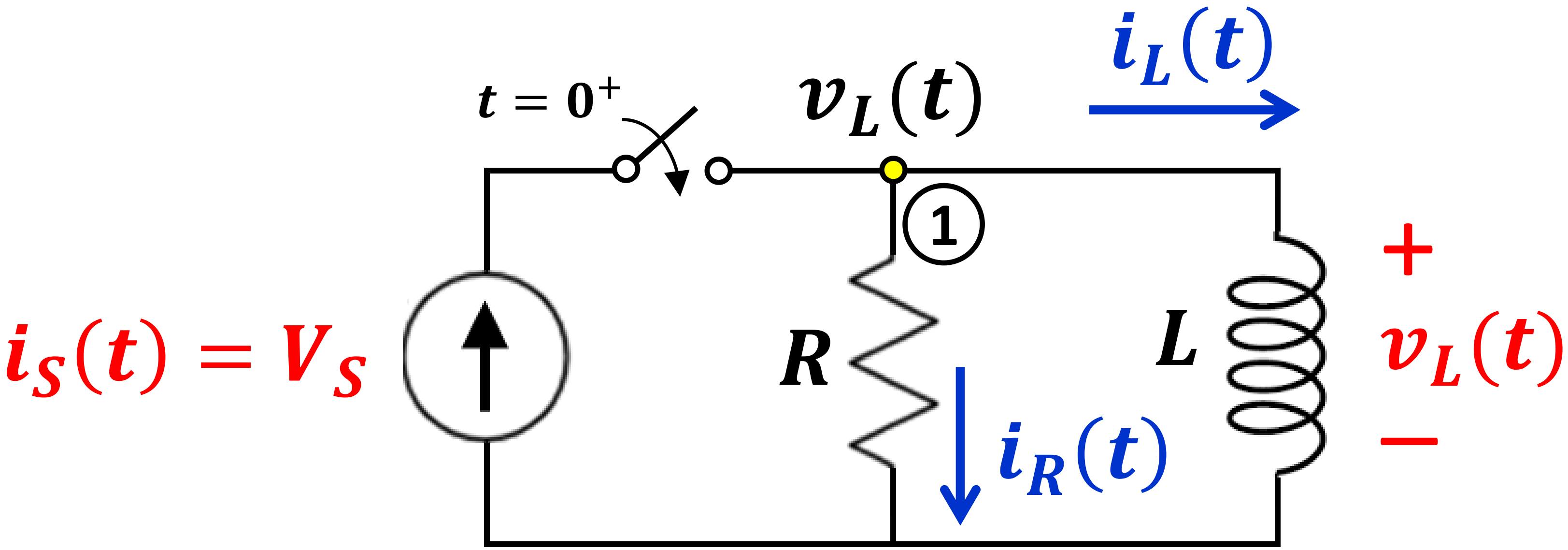
$$V_C(0^-) = V_C(0^+) \rightarrow V_C(0^-) = K_1 + K_2$$

$$\longrightarrow \quad K_1 = V_C(0^-) - K_2$$

Also, $V_C(t \rightarrow \infty) = K_2 = V_S$

$$V_c(t) = [V_C(0^-) - V_S]\, e^{-t/RC} + V_S$$

CASE 2 – RL circuit with constant $i_S(t) = V_S$



Following the same solution steps

$$i_L(t) = [I_L(0^-) - I_S]\, e^{-(L/R)t} + I_S$$

CASE 3 – RC or RL circuit with arbitrary input

Differential equation of the type

$$\frac{d}{dt}y(t) + a\,y(t) = b\,f(t)$$

General solution

$$y(t) = y_h(t) + y_p(t) \qquad t \geq 0$$

Solution of the homogeneous equation

Particular solution

$$\frac{d}{dt}y(t) + a\,y(t) = 0$$

Solution to the homogeneous equation has the form

$$y_h(t) = K_1 e^{-\alpha t}$$

so, the general solution is

$$y_h(t) = \underbrace{K_1 e^{-\alpha t}}_{\text{Homogeneous solution}} + \underbrace{y_p(t)}_{\text{Particular solution}}$$

Homogeneous solution

Particular solution

The particular solution depends on the form of the forcing term $f(t)$ (the source).

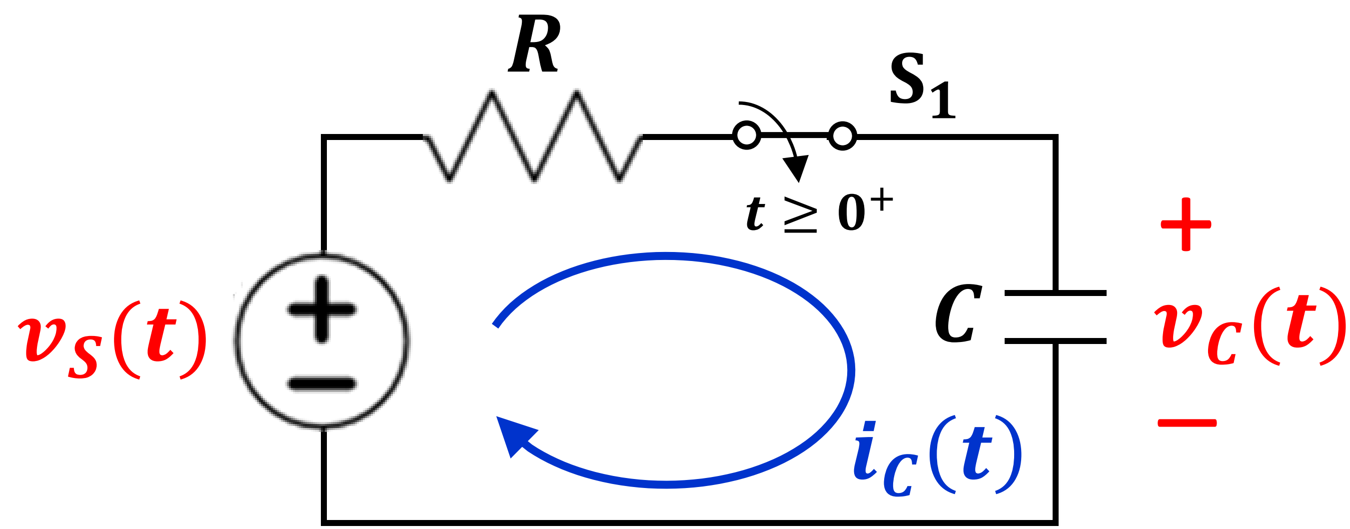
Examples of particular solutions to

$$\frac{d}{dt}y(t) + a\,y(t) = b\,f(t)$$

	function $f(t)$	Particular solution $y_p(t)$
1	constant D	constant K
2	$D \times t$	$Kt + L$ for some K and L
3	$D \times e^{mt}$	$K\,e^{mt}$ if $m \neq -a$ $K\,te^{mt}$ if $m = -a$
4	$\cos(\omega t)$	$A\cos(\omega t + \theta)$ *
5	$\sin(\omega t)$	$A\sin(\omega t + \theta)$ *

* A and θ depend on ω, a, and b

Example



For simplicity $RC = 1\text{s}$

$v_S(t) = \cos(t)$

$v_C(t = 0^-) = 0\text{V}$

Differential equation:

$$\frac{1}{RC} = 1\ [\text{s}^{-1}]$$

$$\frac{d}{dt} v_C(t) + v_C(t) = \cos(\,t)$$

$\omega = 2\pi f = 1$

$$\frac{d}{dt}v_C(t) + v_C(t) = \cos(t)$$

From the table of particular solutions

$$V_p(t) = \mathrm{A}\cos(\mathrm{t} + \theta)$$

$$= \underbrace{A\cos(\theta)}_{B}\cos(t) - \underbrace{A\sin(\theta)}_{C}\sin(t)$$

$$V_p(t) = B\cos(t) - C\sin(t)$$

$$\frac{d}{dt}V_p(t) = -B\sin(t) - C\cos(t)$$

Substituting the particular solution and its derivative

$$V_p(t) = B\cos(t) - C\sin(t)$$

$$\frac{d}{dt}V_p(t) = -B\sin(t) - C\cos(t)$$

in the original differential equation

$$\frac{d}{dt}v_C(t) + v_C(t) = \cos(t)$$

$$\Rightarrow \quad -B\cos(t) - C\sin(t) + B\sin(t) - C\cos(t) = \cos(t)$$

$$(B - C)\cos(t) - (B + C)\sin(t) = \cos(t)$$

$$(B - C) = 1 \qquad (B + C) = 0$$

$$(B - C) = 1 \qquad (B + C) = 0$$

$$B = A\cos\theta = \frac{1}{2}$$

$$C = A\sin(\theta) = -\frac{1}{2}$$

From the ratio C/B

$$\frac{C}{B} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta) = -1$$

we obtain readily that

$$\theta = -\frac{\pi}{4}$$

From above:

$$A = \frac{1}{2\cos\theta} = \frac{1}{2\cos\left(-\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}$$

From these results, the particular solution is

$$V_p(t) = \mathrm{A}\cos(\mathrm{t}+\theta) = \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$$

and the complete solution is

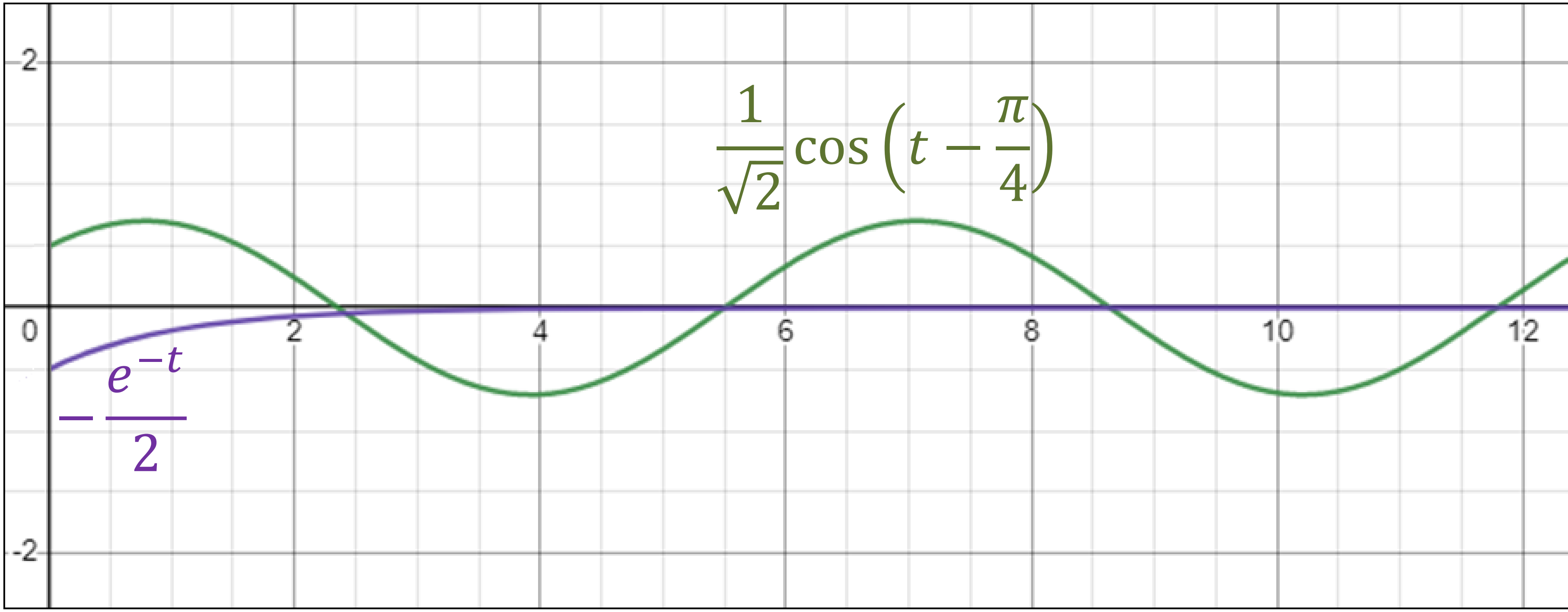
$$V_C(t) = K_1 e^{-t} + \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$$

At $t = 0^+$

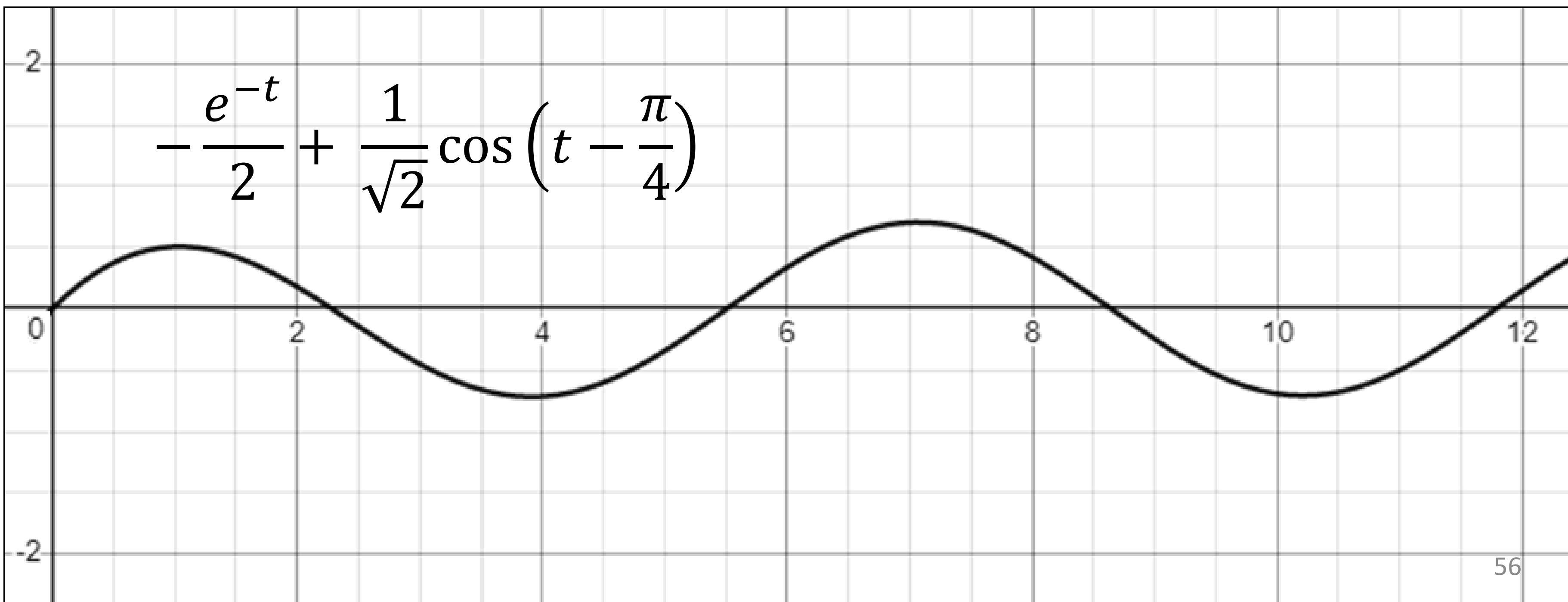
$$0 = K_1 + \frac{1}{\sqrt{2}}\cos\left(-\frac{\pi}{4}\right) = K_1 + \frac{1}{2} \Rightarrow K_1 = -\frac{1}{2}$$

Finally:

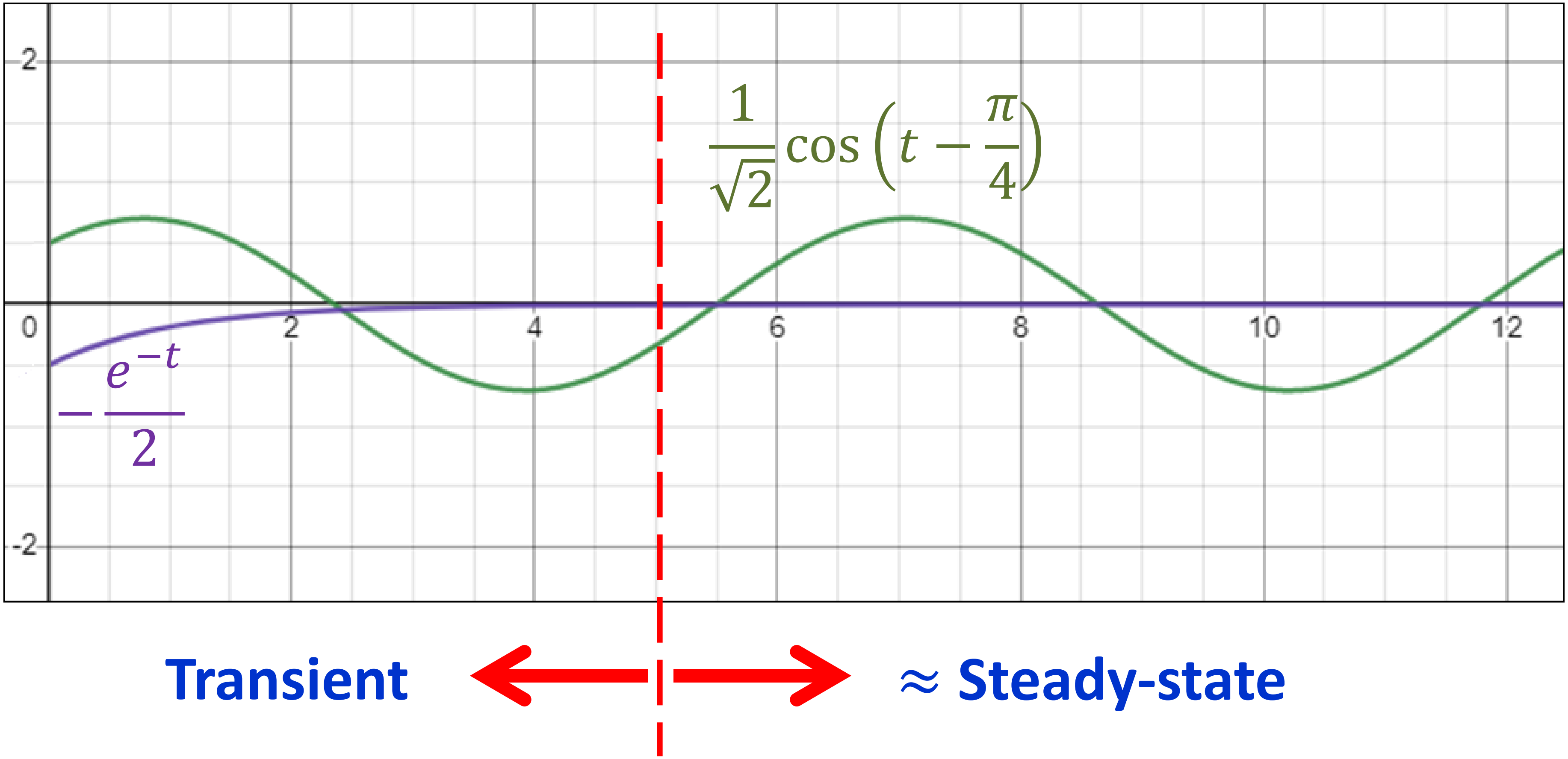
$$V_C(t) = -\frac{e^{-t}}{2} + \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$$



$\frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$
$-\frac{e^{-t}}{2}$
2
-2
0
2
4
6
8
10
12



$-\frac{e^{-t}}{2}+\frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$
2
-2
0
2
4
6
8
10
12



In many practical situations the transients are very short. We are interested in describing steady-state system response with a more immediate mathematical approach for a specified frequency of operation. This will be accomplished next by introducing the phasor formalism.