ECE 205 “Electrical and Electronics Circuits”

Spring 2024 – LECTURE 16

MWF – 12:00pm

Prof. Umberto Ravaioli

2062 ECE Building

Lecture 16 – Summary

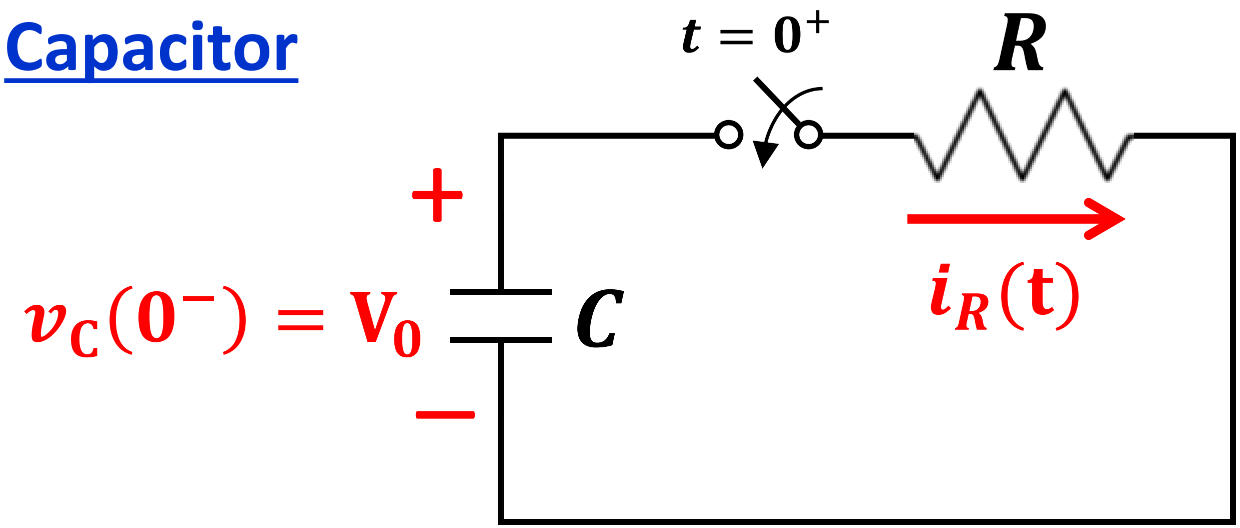
Learning Objectives

1. Power considerations
2. Detailed analysis of RC and RL circuits with time-dependent sources
3. Solution of circuit differential equations

Power Considerations

1. **Energy stored by a capacitor**
2. **Energy stored by an inductor**

Capacitor



$$\alpha = \frac{1}{\tau} = \frac{1}{RC}$$

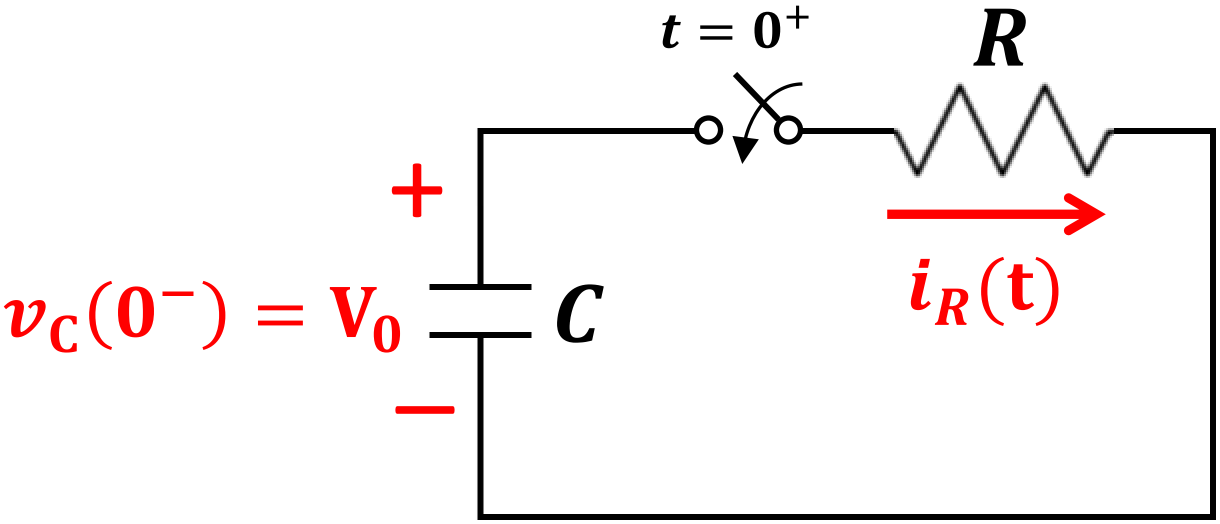
$$C = \frac{\tau}{R}$$

At $t \to \infty$ the resistor has dissipated all the energy stored in the capacitor

Power $$p_R(t) = v_C(t) \cdot i_R(t) = \frac{v_c^2(t)}{R} = \frac{V_0^2 e^{-2\alpha t}}{R}$$

Energy absorbed $$w_R(t) = \int_0^t p_R(t')\, dt' = \int_0^t \frac{V_0^2 e^{-2\alpha t'}}{R}\, dt'$$

$$= -\frac{\tau V_0^2}{2R} e^{-2\alpha t}\Big|_0^t = \frac{1}{2} C\, V_0^2 \left[1 - e^{-2\alpha t}\right]$$



$$\alpha = \frac{1}{\tau} = \frac{1}{RC}$$

$$C = \frac{\tau}{R}$$

Energy absorbed

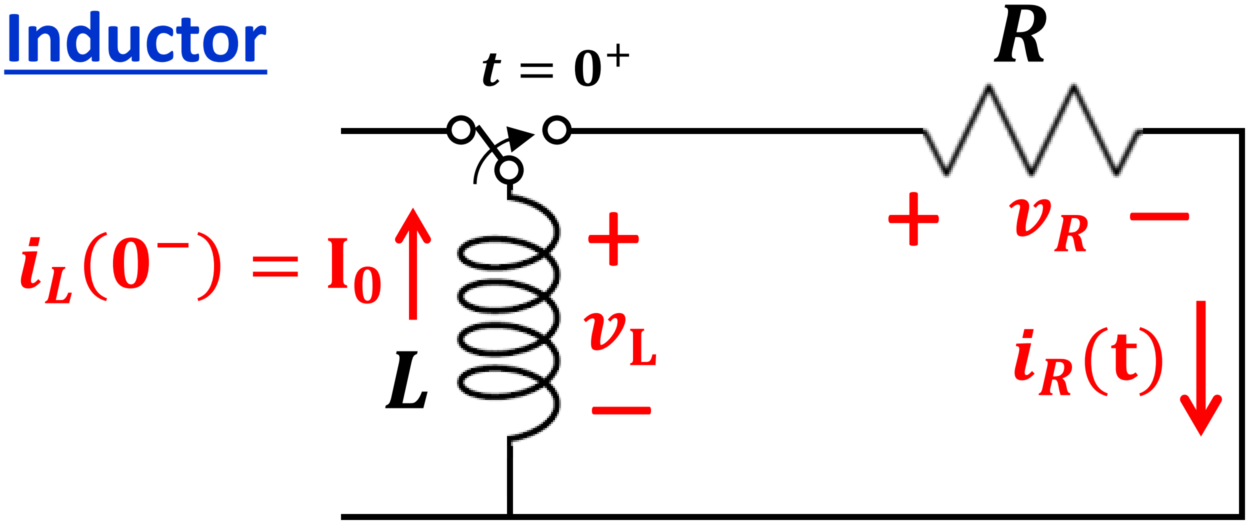
$$w_R(t) = \frac{1}{2} C\, V_0^2 \left[1 - e^{-2\alpha t}\right]$$

$t \rightarrow \infty$

$$w_R(t \rightarrow \infty) = \frac{1}{2} C\, V_0^2$$

Energy dissipated by R = Energy stored by C

Inductor



$$\alpha = \frac{1}{\tau} = \frac{R}{L}$$

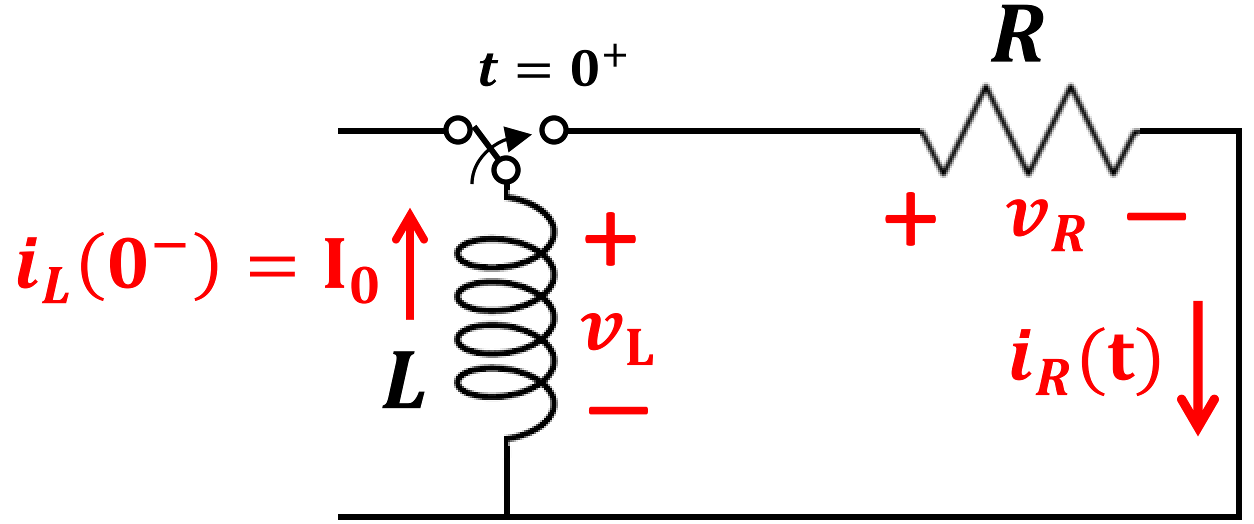
$$L = \tau R$$

At $t \to \infty$ the resistor has dissipated all the energy stored in the inductor

Power $$p_R(t) = v_R(t) \cdot i_R(t) = i_L^2(t)R = RI_0^2 e^{-2\alpha t}$$

Energy absorbed $$w_R(t) = \int_0^t p_R(t')\,dt' = \int_0^t RI_0^2 e^{-2\alpha t'}\,dt'$$

$$= -\frac{\tau R I_0^2}{2} e^{-2\alpha t}\Big|_0^t = \frac{1}{2} L\, I_0^2\left[1 - e^{-2\alpha t}\right]$$



$$\alpha = \frac{1}{\tau} = \frac{R}{L}$$

$$L = \tau R$$

Energy absorbed

$$w_R(t) = \frac{1}{2} L\, I_0^2 \left[1 - e^{-2\alpha t}\right]$$

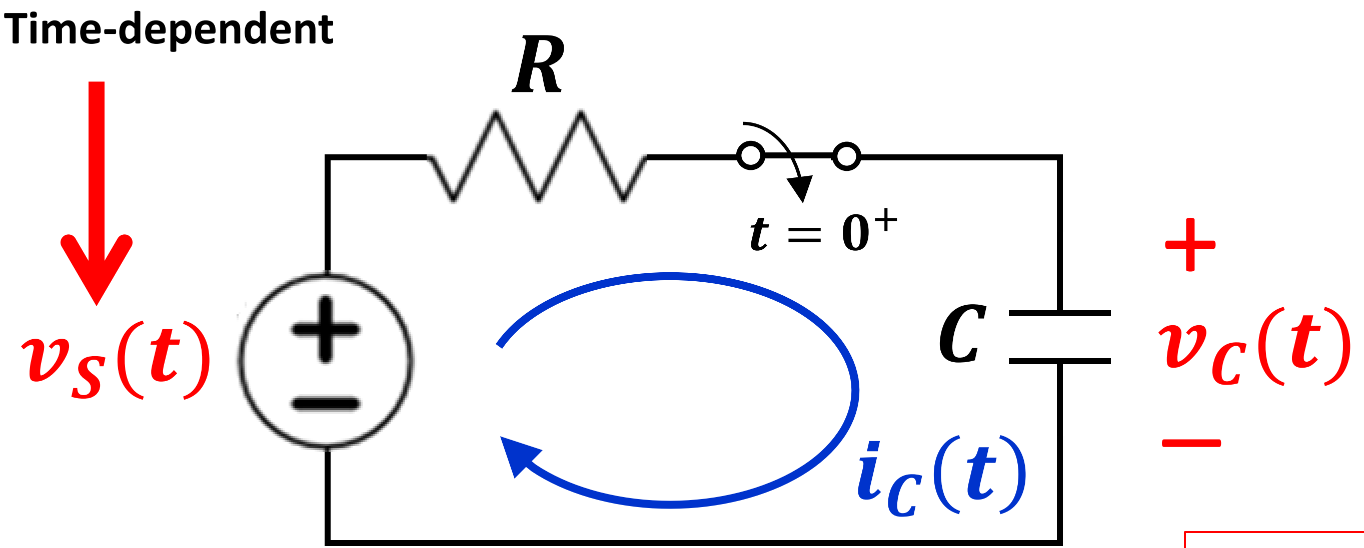
$t \to \infty$

$$w_R(t \to \infty) = \frac{1}{2} L\, I_0^2$$

Energy dissipated by R = Energy stored by L

RC and RL circuits with time-dependent input

RC circuit



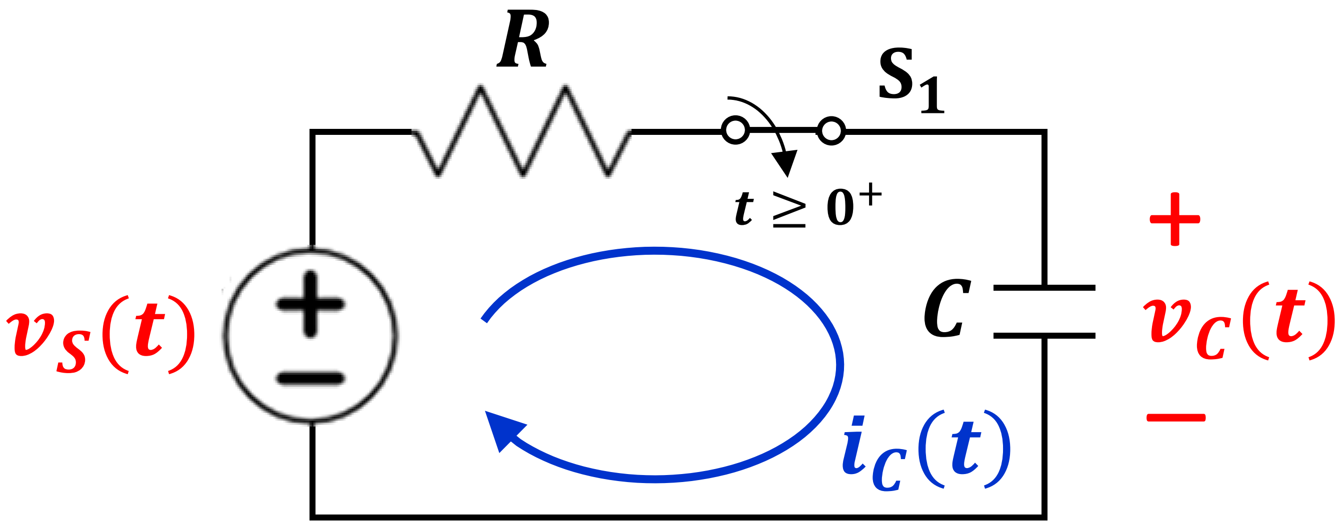
$$i_C(t) = C\frac{dv_C(t)}{dt}$$

KVL for the RC circuit

$$-v_S(t) + Ri_C(t) + v_C(t) = 0$$

$$RC\frac{d}{dt}v_C(t) + v_C(t) = v_S(t)$$

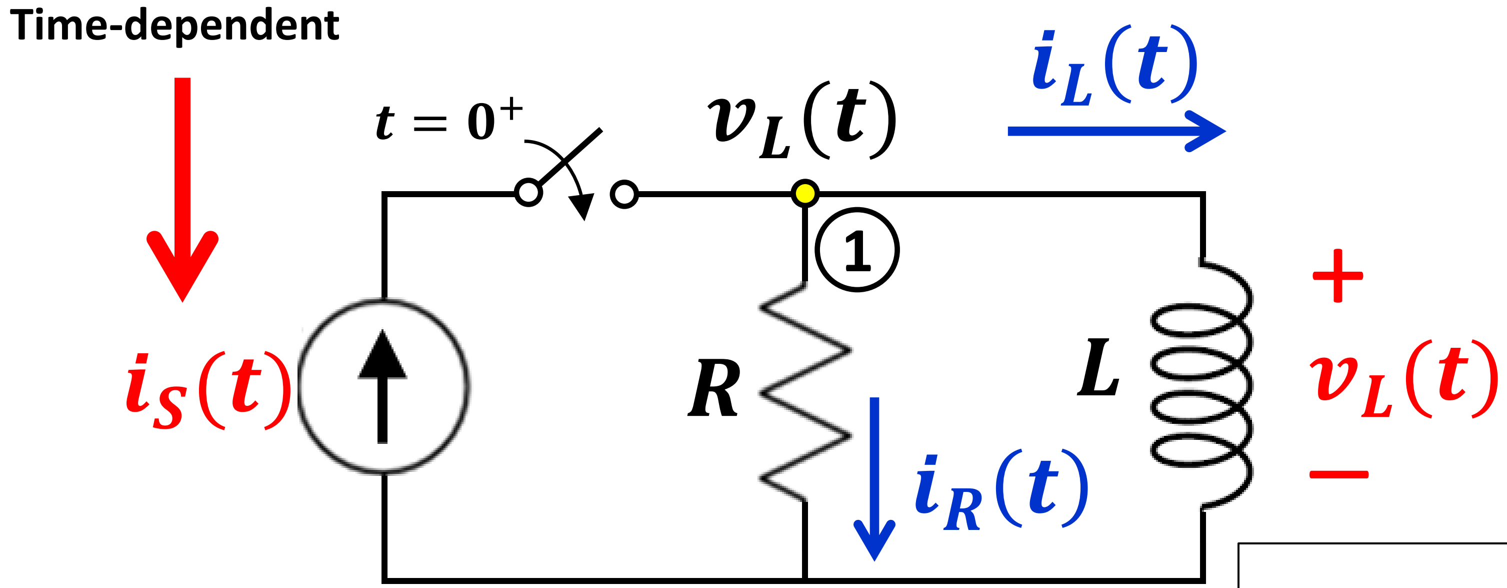
RC circuit



The RC circuit is described by the differential equation

$$\frac{d}{dt} v_C(t) + \frac{1}{RC} v_C(t) = \frac{1}{RC} v_S(t)$$

RL circuit



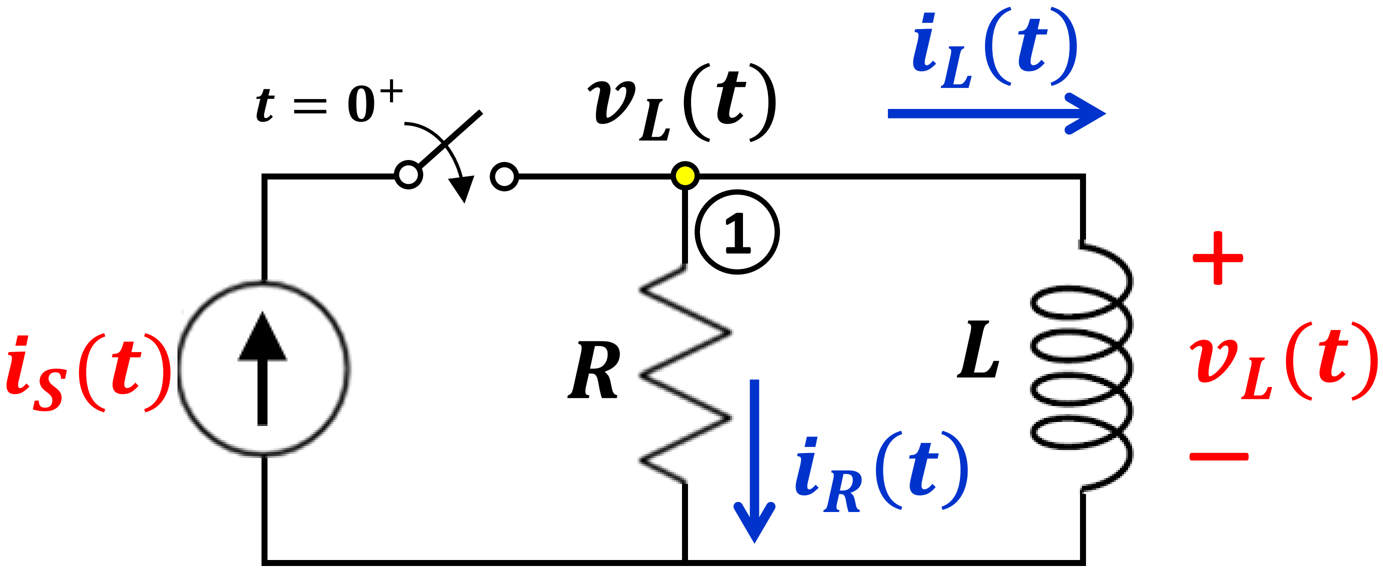
$$v_L(t) = L\frac{d}{dt}i_L(t)$$

KVL for the RL loop

$$v_L(t) - R[i_S(t) - i_L(t)] = 0$$

$$L\frac{d}{dt}i_L(t) + Ri_L(t) = Ri_S(t)$$

RL circuit



The RL circuit is described by the differential equation

$$\frac{d}{dt} i_L(t) + \frac{R}{L} i_L(t) = \frac{R}{L} i_S(t)$$

The transient behavior for RC and RL circuits is described by the first order ordinary differential equation

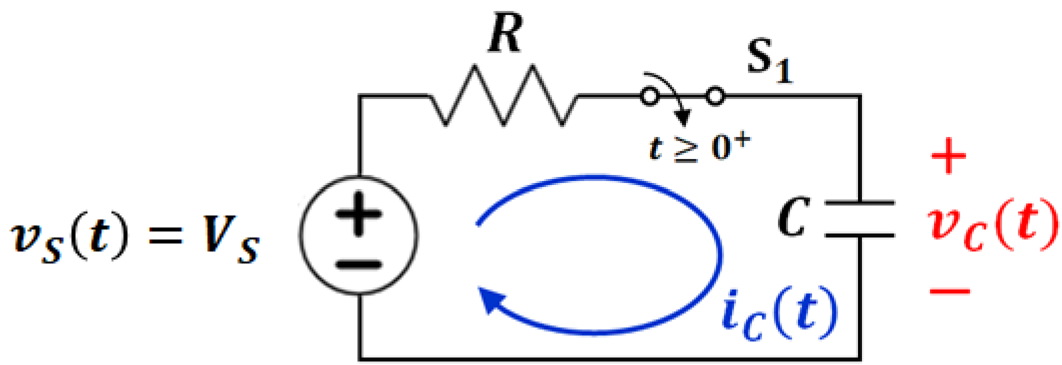
$$\frac{d}{dt}y(t) + a\,y(t) = b\,f(t)$$

Forcing term

A solution satisfies the initial condition

$$y(0^-) = y(0^+)$$

CASE 1 – RC circuit with constant $v_S(t) = V_S$



$$\frac{d}{dt}V_C(t) + \frac{1}{RC}V_C(t) = \frac{1}{RC}V_S \qquad \boxed{1}$$

Initial condition

$$V_C(0^-) = V_C(0^+)$$

The solution can be written in the form

$$V_C(t) = u_h(t) + \underbrace{K_2}_{\text{constant}}$$

In our case, we have:

$$V_C(t) = V_h(t) + V_S$$

$$\downarrow$$

$$\frac{d}{dt}[V_h + V_S] + \frac{1}{RC}[V_h + \cancel{V_S}] = \frac{1}{RC}\cancel{V_S}$$

$$\downarrow$$

$$\frac{d}{dt}V_h(t) + \frac{1}{RC}V_h(t) = 0 \qquad \boxed{2}$$

Equation 2 is homogeneous and can be integrated as

$$V_h(t) = K_1\, e^{-t/RC}$$

Complete solution

$$V_c(t) = V_h(t) + K_2 = K_1\, e^{-t/RC} + K_2$$

Equation [2] is homogeneous and can be integrated as

$$V_h(t) = K_1\, e^{-t/RC}$$

Complete solution

$$V_c(t) = V_h(t) + K_2 = K_1\, e^{-t/RC} + K_2$$

$$V_C(0^-) = V_C(0^+) \rightarrow V_C(0^-) = K_1 + K_2$$

$$\longrightarrow\; K_1 = V_C(0^-) - K_2$$

Also, $V_C(t \rightarrow \infty) = K_2 = V_S$

Equation $\boxed{2}$ is homogeneous and can be integrated as

$$V_h(t) = K_1\, e^{-t/RC}$$

Complete solution

$$V_c(t) = V_h(t) + K_2 = K_1\, e^{-t/RC} + K_2$$

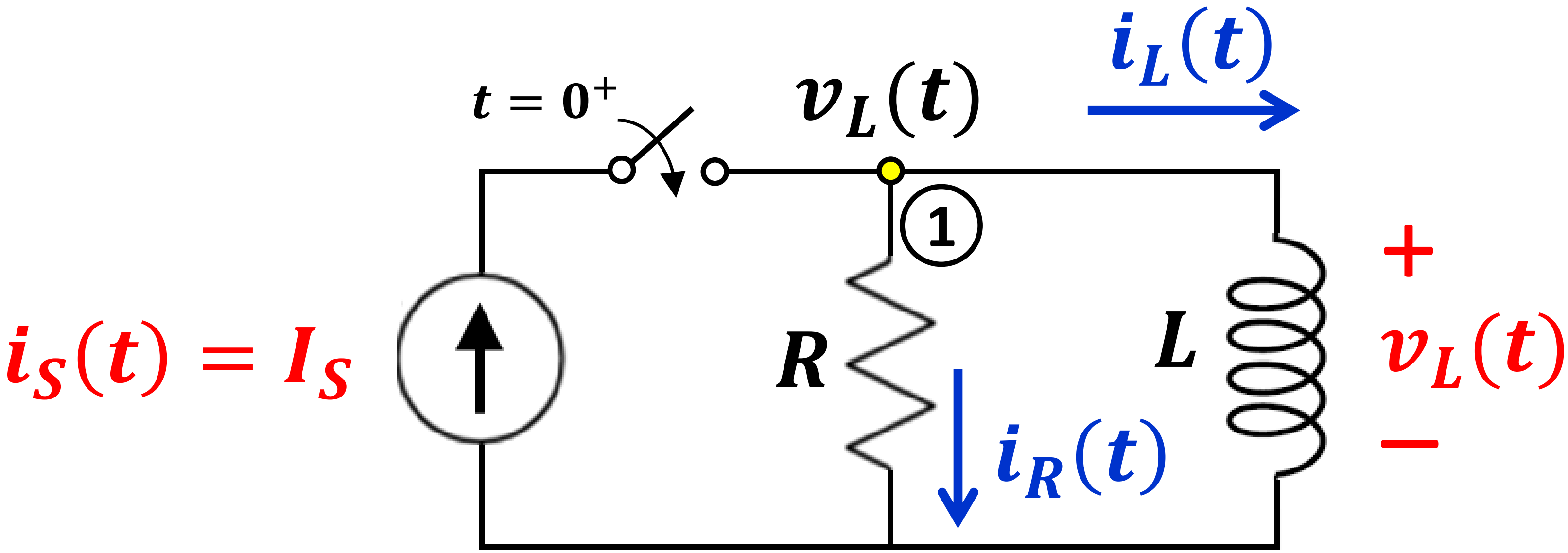
$$V_C(0^-) = V_C(0^+) \;\rightarrow V_C(0^-) = K_1 + K_2$$

$$\longrightarrow \boxed{K_1 = V_C(0^-) - K_2}$$

Also, $\boxed{V_C(t \rightarrow \infty) = K_2 = V_S}$

$$V_c(t) = [V_C(0^-) - V_S]\, e^{-t/RC} + V_S$$

CASE 2 – RL circuit with constant $i_S(t) = I_S$



Following the same solution steps

$$i_L(t) = [I_L(0^-) - I_S]\, e^{-(L/R)t} + I_S$$

CASE 3 – RC or RL circuit with arbitrary input

Differential equation of the type

$$\frac{d}{dt}y(t) + a\,y(t) = b\,f(t)$$

General solution

$$y(t) = y_h(t) + y_p(t) \qquad t \geq 0$$

Solution of the homogeneous equation

Particular solution

$$\frac{d}{dt}y(t) + a\,y(t) = 0$$

Solution to the homogeneous equation has the form

$$y_h(t) = K_1 e^{-\alpha t}$$

so, the general solution is

$$y_h(t) = K_1 e^{-\alpha t} + y_p(t)$$

$\underbrace{K_1 e^{-\alpha t}}_{\text{Homogeneous solution}}$ $\underbrace{y_p(t)}_{\text{Particular solution}}$

Homogeneous solution

Particular solution

The particular solution depends on the form of the forcing term $f(t)$ (the source).

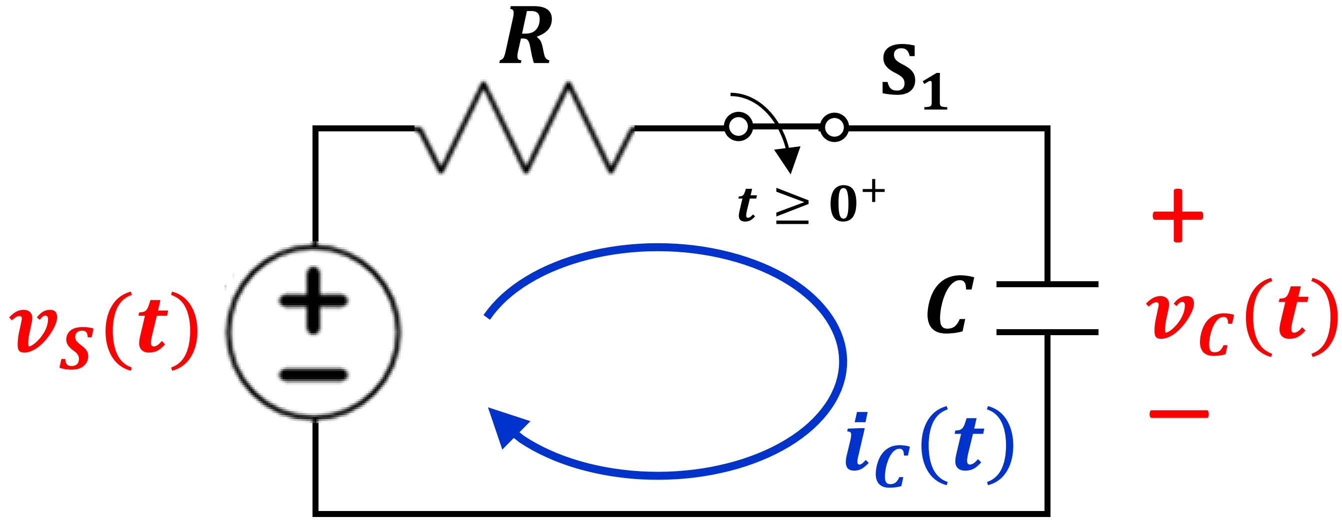
Examples of particular solutions to

$$\frac{d}{dt} y(t) + a\, y(t) = b\, f(t)$$

	function $f(t)$	Particular solution $y_p(t)$
1	constant D	constant K
2	$D \times t$	$Kt + L$ for some K and L
3	$D \times e^{mt}$	$K\, e^{mt}$ if $m \neq -a$ $K\, te^{mt}$ if $m = -a$
4	$\cos(\omega t)$	$A \cos(\omega t + \theta)$ *
5	$\sin(\omega t)$	$A \sin(\omega t + \theta)$ *

* A and θ depend on $\omega, a,$ and b

Example



For simplicity $RC = 1\text{s}$

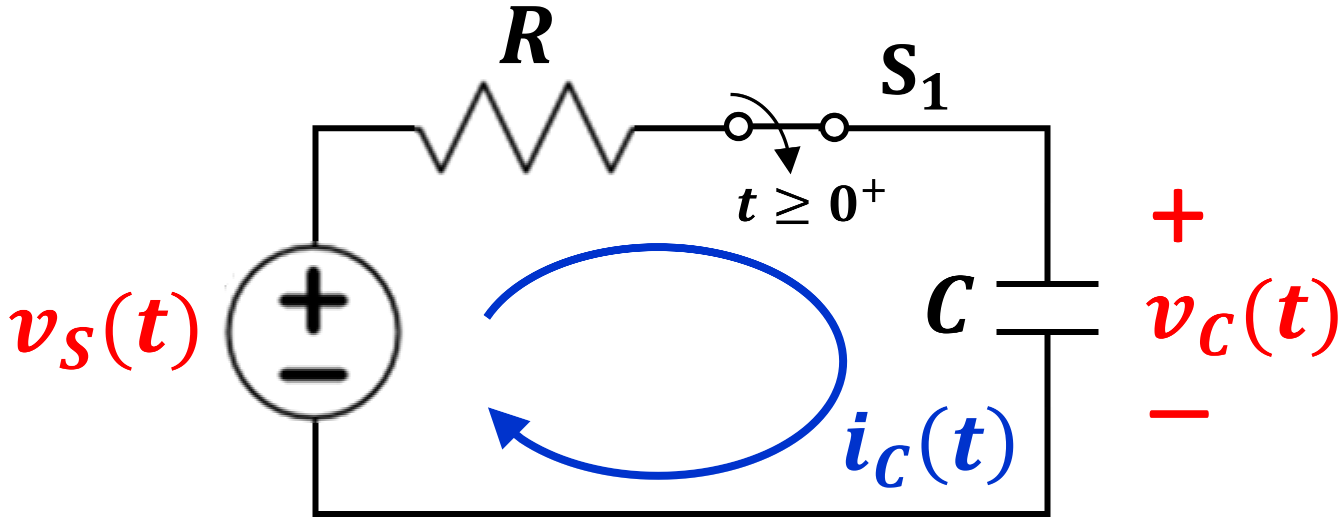
$$v_S(t) = \cos(t)$$

$$v_C(t = 0^-) = 0\text{V}$$

zero initial condition

(the capacitor is not charged at time $t = 0^-$)

Example



For simplicity $RC = 1\text{s}$

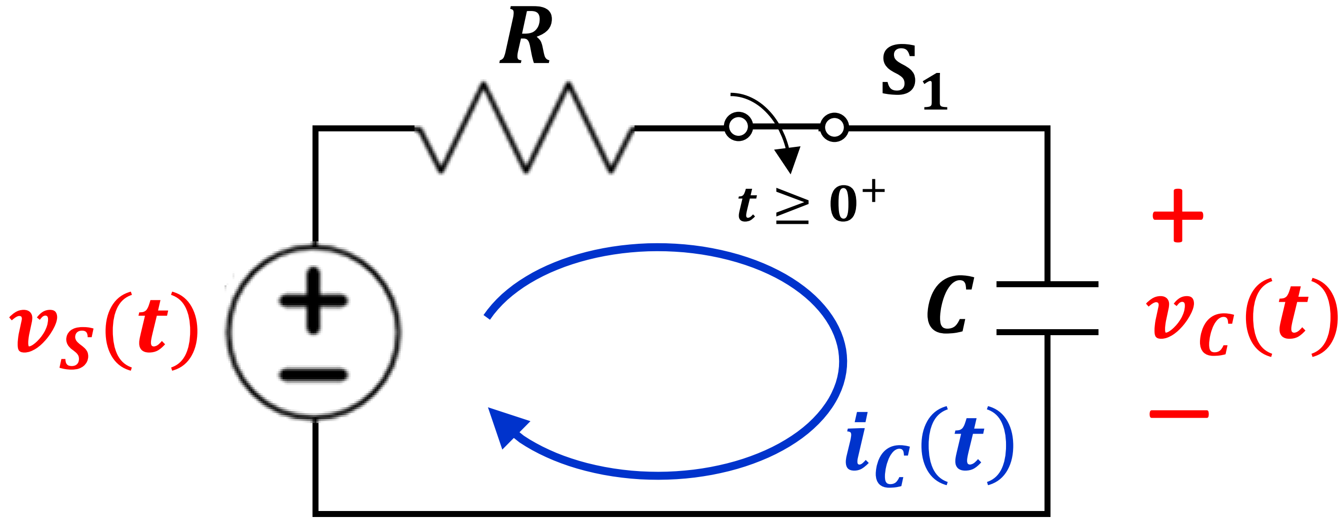
$v_S(t) = \cos(t)$

$v_C(t = 0^-) = 0\text{V}$

Differential equation:

$$\frac{d}{dt} v_C(t) + v_C(t) = \cos(t)$$

Example



For simplicity $RC = 1\text{s}$

$v_S(t) = \cos(t)$

$v_C(t = 0^-) = 0\text{V}$

Differential equation:

$$\frac{1}{RC} = 1\ [\text{s}^{-1}]$$

$$\frac{d}{dt} v_C(t) + v_C(t) = \cos(\ t)$$

$\omega = 2\pi f = 1$

$$\frac{d}{dt}v_C(t) + v_C(t) = \cos(t)$$

From the table of particular solutions

$$V_p(t) = \mathrm{A}\cos(\mathrm{t} + \theta)$$

$$= \underbrace{A\cos(\theta)}_{\boldsymbol{B}}\cos(t) - \underbrace{A\sin(\theta)}_{\boldsymbol{C}}\sin(t)$$

$$\frac{d}{dt}v_C(t) + v_C(t) = \cos(t)$$

From the table of particular solutions

$$V_p(t) = \mathrm{A}\cos(\mathrm{t} + \theta)$$

$$= \underbrace{A\cos(\theta)}_{B}\cos(t) - \underbrace{A\sin(\theta)}_{C}\sin(t)$$

$$V_p(t) = B\cos(t) - C\sin(t)$$

$$\frac{d}{dt}V_p(t) = -B\sin(t) - C\cos(t)$$

Substituting the particular solution and its derivative

$$V_p(t) = B\cos(t) - C\sin(t)$$

$$\frac{d}{dt}V_p(t) = -B\sin(t) - C\cos(t)$$

in the original differential equation

$$\frac{d}{dt}v_C(t) + v_C(t) = \cos(t)$$

Substituting the particular solution and its derivative

$$V_p(t) = B\cos(t) - C\sin(t)$$

$$\frac{d}{dt}V_p(t) = -B\sin(t) - C\cos(t)$$

in the original differential equation

$$\Rightarrow \quad \frac{d}{dt}V_p(t) + V_p(t) = \cos(t)$$

Substituting the particular solution and its derivative

$$V_p(t) = B\cos(t) - C\sin(t)$$

$$\frac{d}{dt}V_p(t) = -B\sin(t) - C\cos(t)$$

in the original differential equation

$$\frac{d}{dt}V_p(t) + V_p(t) = \cos(t)$$

$$\Rightarrow \quad -B\sin(t) - C\cos(t) \\ +B\cos(t) - C\sin(t) = \cos(t)$$

Substituting the particular solution and its derivative

$$V_p(t) = B\cos(t) - C\sin(t)$$

$$\frac{d}{dt}V_p(t) = -B\sin(t) - C\cos(t)$$

in the original differential equation

$$\frac{d}{dt}V_p(t) + V_p(t) = \cos(t)$$

$$\Rightarrow \quad -B\sin(t) - C\cos(t) + B\cos(t) - C\sin(t) = \cos(t)$$

$$(B - C)\cos(t) - (B + C)\sin(t) = \cos(t)$$

$$(B - C) = 1 \qquad (B + C) = 0$$

$$(B - C) = 1 \qquad (B + C) = 0$$

$$\Rightarrow \quad B = \frac{1}{2} = A\cos\theta \qquad C = -\frac{1}{2} = A\sin(\theta)$$

definitions of B and C

$$(B - C) = 1 \qquad (B + C) = 0$$

$$\Rightarrow \boxed{B = \frac{1}{2} = A\cos\theta} \qquad \boxed{C = -\frac{1}{2} = A\sin(\theta)}$$

definitions of B and C

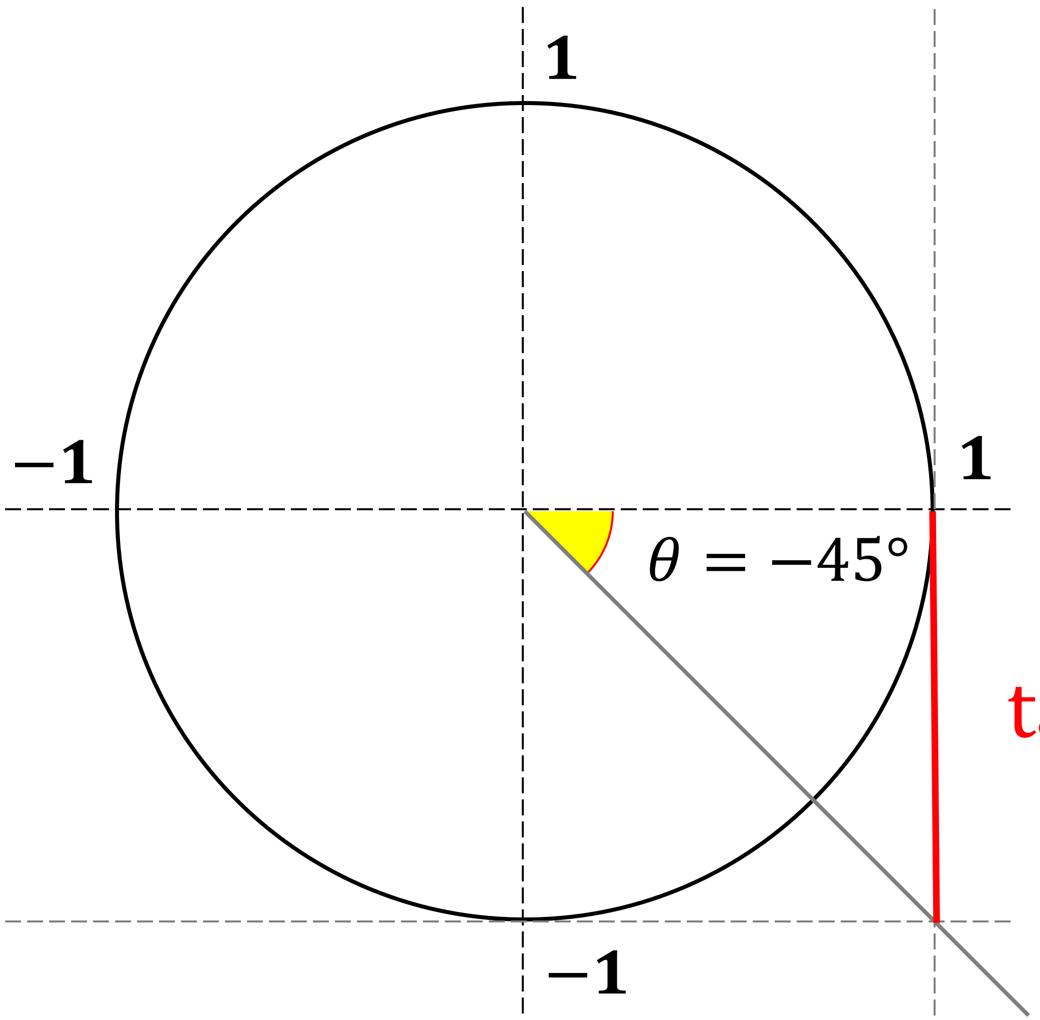
From the ratio C/B

$$\frac{C}{B} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta) = -1$$

we obtain readily that

$$\Rightarrow \boxed{\theta = -\frac{\pi}{4}}$$

$$\theta = -\frac{\pi}{4}$$



$$B = A\cos\theta = \frac{1}{2}$$

$$C = A\sin(\theta) = -\frac{1}{2}$$

$$B = A\cos\theta = \frac{1}{2}$$

$$C = A\sin(\theta) = -\frac{1}{2}$$

$$A = \frac{1}{2\cos\theta} = \frac{1}{2\cos\left(-\frac{\pi}{4}\right)} = \frac{1}{2\frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{2}}$$

$$B = A\cos\theta = \frac{1}{2}$$

$$C = A\sin(\theta) = -\frac{1}{2}$$

$$A = \frac{1}{2\cos\theta} = \frac{1}{2\cos\left(-\frac{\pi}{4}\right)} = \frac{1}{2\frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{2}}$$

$$A = \frac{-1}{2\sin\theta} = \frac{-1}{2\sin\left(-\frac{\pi}{4}\right)} = \frac{-1}{2\frac{-\sqrt{2}}{2}} = \frac{1}{\sqrt{2}}$$

$$B = A\cos\theta = \frac{1}{2}$$

$$C = A\sin(\theta) = -\frac{1}{2}$$

$$A = \frac{1}{2\cos\theta} = \frac{1}{2\cos\left(-\frac{\pi}{4}\right)} = \frac{1}{2\frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{2}}$$

$$A = \frac{-1}{2\sin\theta} = \frac{-1}{2\sin\left(-\frac{\pi}{4}\right)} = \frac{-1}{2\frac{-\sqrt{2}}{2}} = \frac{1}{\sqrt{2}}$$

$$A = \frac{1}{\sqrt{2}}$$

From these results, the particular solution is

$$V_p(t) = \mathrm{A}\cos(\mathrm{t} + \theta) = \frac{1}{\sqrt{2}}\cos\left(t - \frac{\pi}{4}\right)$$

and the complete solution is

$$V_C(t) = K_1 e^{-t} + \frac{1}{\sqrt{2}}\cos\left(t - \frac{\pi}{4}\right)$$

From these results, the particular solution is

$$V_p(t) = \mathrm{A}\cos(\mathrm{t}+\theta) = \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$$

and the complete solution is

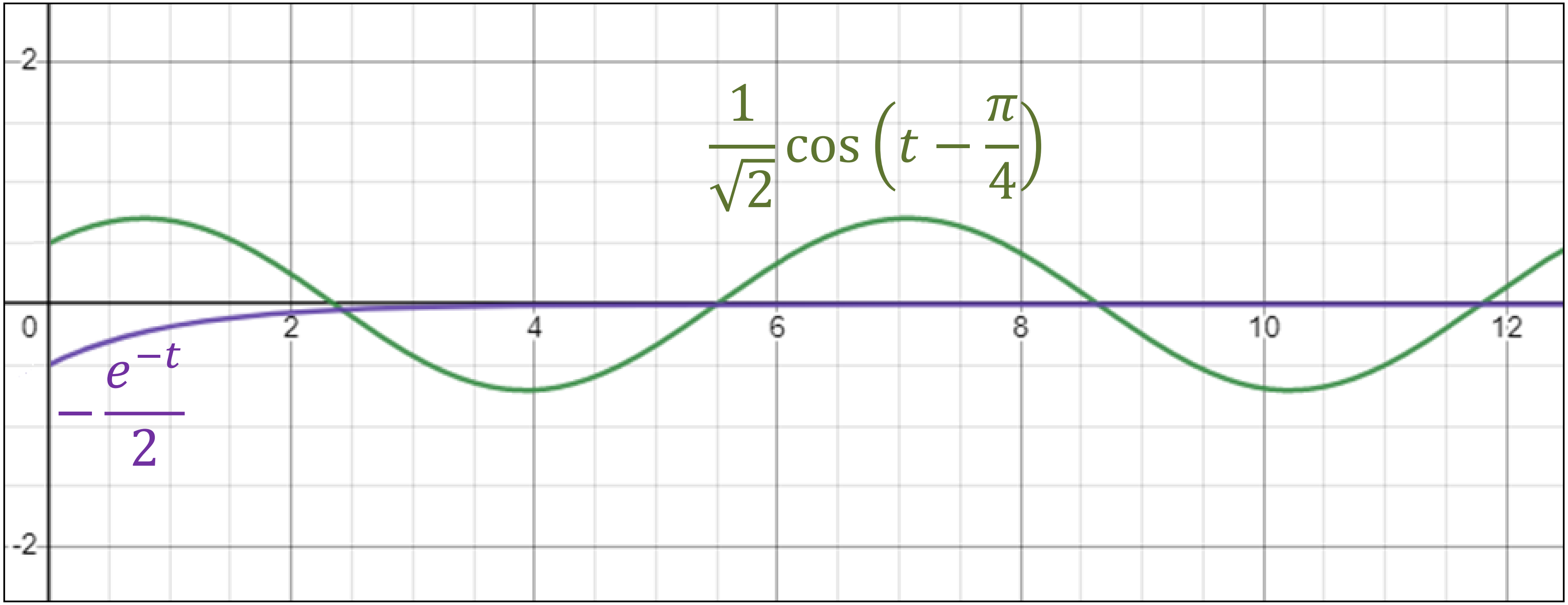
$$V_C(t) = K_1 e^{-t} + \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$$

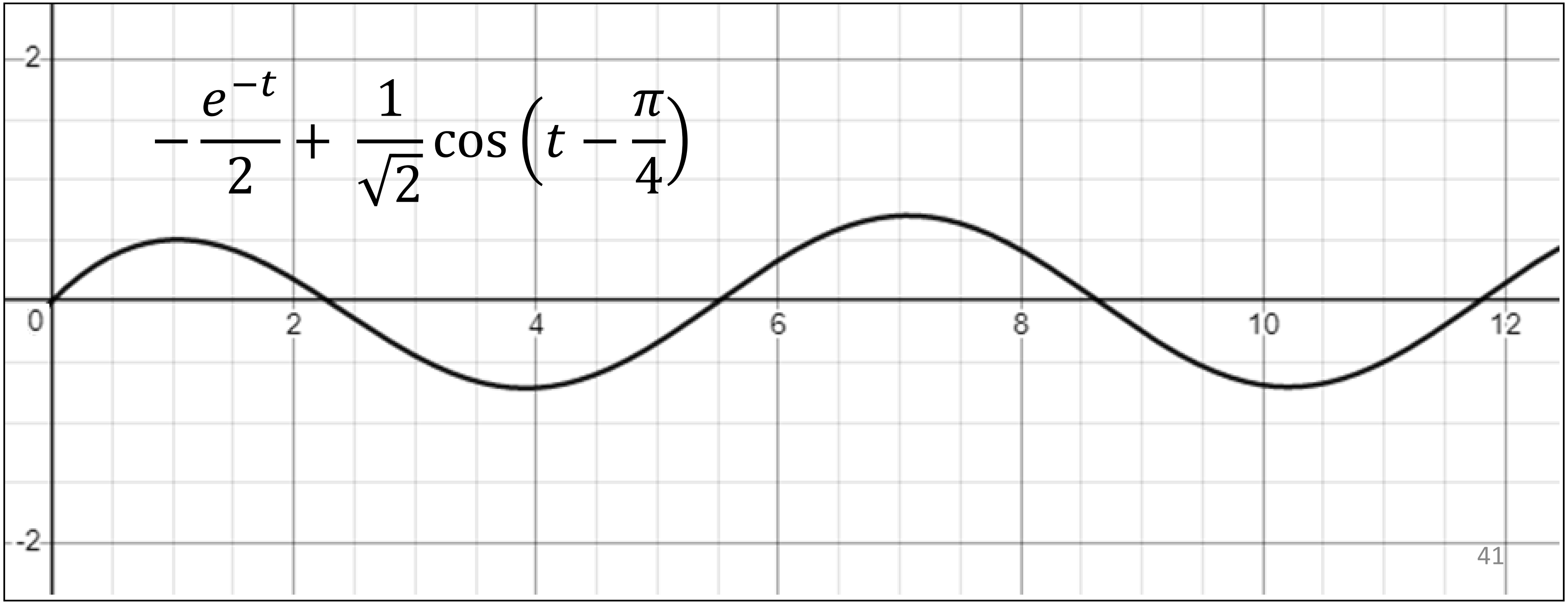
At $t = 0^+$

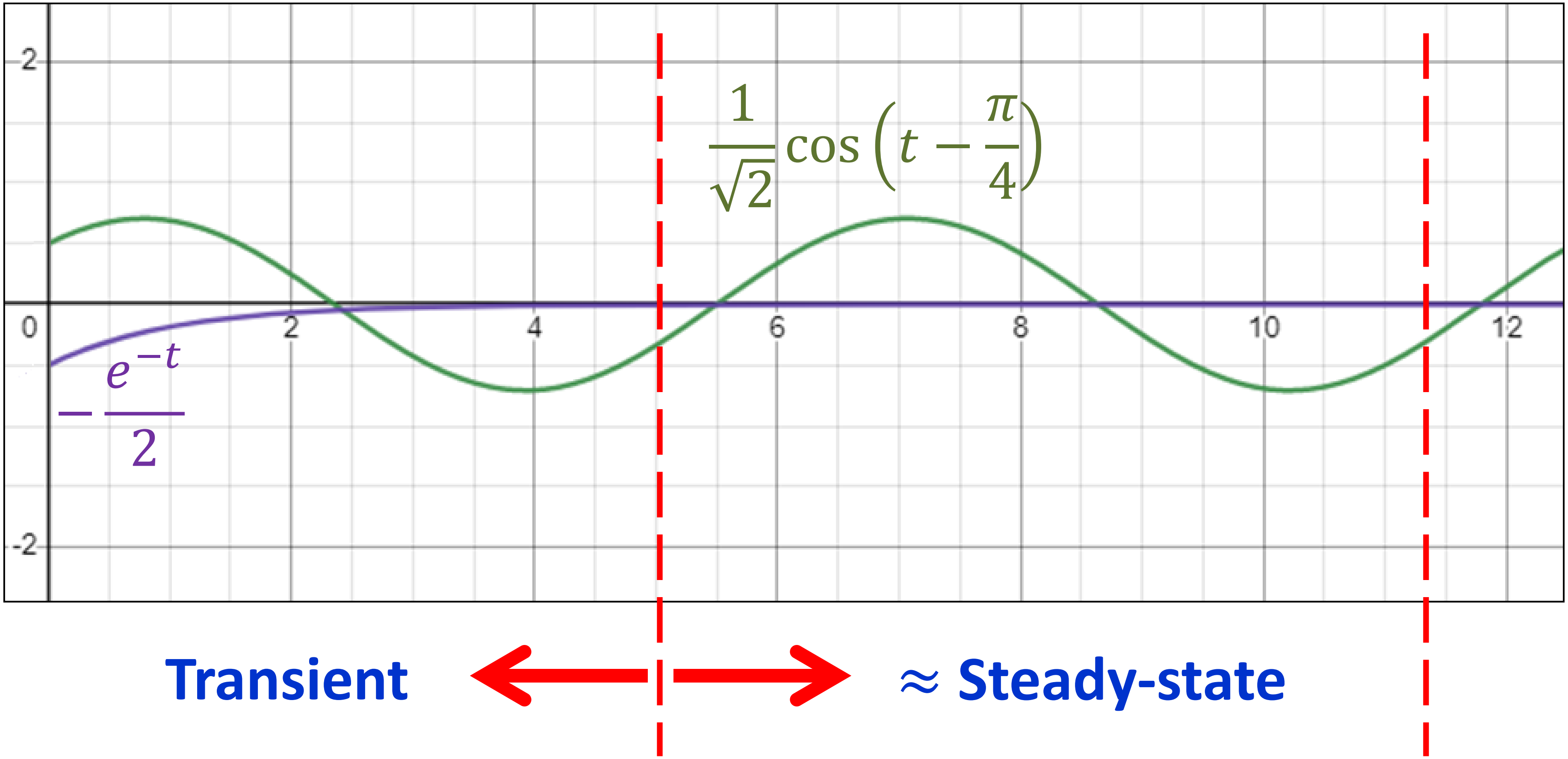
$$0 = K_1 + \frac{1}{\sqrt{2}}\cos\left(-\frac{\pi}{4}\right) = K_1 + \frac{1}{2} \Rightarrow K_1 = -\frac{1}{2}$$

Finally:

$$V_C(t) = -\frac{e^{-t}}{2} + \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$$


$\frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$
$-\frac{e^{-t}}{2}$
2
-2
0
2
4
6
8
10
12


$-\frac{e^{-t}}{2}+\frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$
2
-2
0
2
4
6
8
10
12



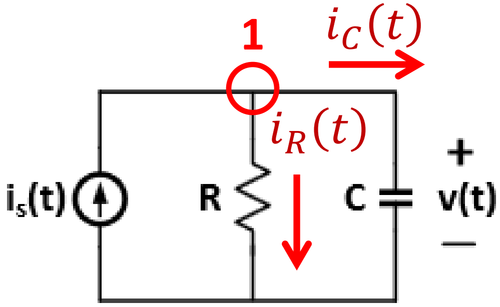
In many practical situations the transients are very short. We are interested in describing steady-state system response with a more immediate mathematical approach for a specified frequency of operation. This will be accomplished next by introducing the phasor formalism.

ECE 205 - HW 7
Written

Name: **UIN:**

1. Using KCL and the v-i relations for resistors and capacitors, show that the voltage $v(t)$ in the following circuit satisfies the following ODE if $R = 5\Omega$ and $C = 2F$. Also find the capacitor voltage $\mathbf{v(t)}$ if $i_s(t) = cos(t)$. Assume zero initial conditions.

$$2\frac{dv}{dt} + \frac{1}{5}v(t) = i_s(t) \text{ for } t > 0.$$



Hints: **Write the KCL at node 1 to verify the given differential equation**

Rewrite the equation in the form $$\frac{d}{dt}v(t) + a\,v(t) = b\,f(t)$$

and follow the procedure we outlined today to find $v(t)$