ECE 205 “Electrical and Electronics Circuits”

Spring 2024 – LECTURE 17

MWF – 12:00pm

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Lecture 17 – Summary

Learning Objectives

1. Review of Complex Numbers
2. Complex number representations
3. Operations with complex numbers

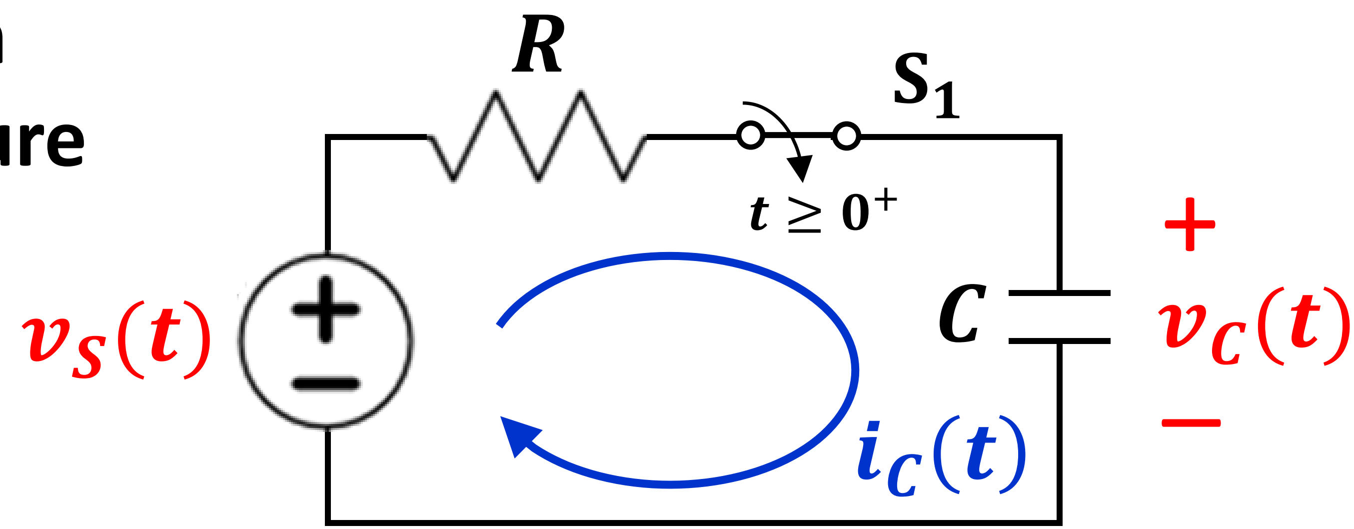
Upcoming Quiz 2

Four Problems

1. **Two problems on Circuits with Dependent Sources**
2. **Two problems on Transient in RC and RL circuits**

Help sheet available at the exam as in Practice Quiz 2

Example from previous lecture



For simplicity $RC = 1\text{s}$

$v_S(t) = \cos(t)$

$v_C(t = 0^-) = 0\text{V}$

$$\frac{1}{RC} = 1\ [\text{s}^{-1}]$$

Differential equation:

$$\frac{d}{dt} v_C(t) + v_C(t) = \cos(t)$$

$\omega = 2\pi f = 1$

$$\frac{d}{dt}v_C(t) + v_C(t) = \cos(t)$$

From the table of particular solutions

$$V_p(t) = \mathrm{A}\cos(\mathrm{t} + \theta)$$

$$= \underbrace{A\cos(\theta)}_{\boldsymbol{B}}\cos(t) - \underbrace{A\sin(\theta)}_{\boldsymbol{C}}\sin(t)$$

$$V_p(t) = B\cos(t) - C\sin(t)$$

$$\frac{d}{dt}V_p(t) = -B\sin(t) - C\cos(t)$$

Substituting the particular solution and its derivative

$$V_p(t) = B\cos(t) - C\sin(t)$$

$$\frac{d}{dt}V_p(t) = -B\sin(t) - C\cos(t)$$

in the original differential equation

$$\frac{d}{dt}v_C(t) + v_C(t) = \cos(t)$$

$$-B\sin(t) - C\cos(t)$$
$$+B\cos(t) - C\sin(t) = \cos(t)$$

$$(B - C)\cos(t) - (B + C)\sin(t) = \cos(t)$$

$$(B - C) = 1 \qquad (B + C) = 0$$

$$(B - C) = 1 \qquad (B + C) = 0$$

$$B = A\cos\theta = \frac{1}{2}$$

$$C = A\sin(\theta) = -\frac{1}{2}$$

From the ratio C/B

$$\frac{C}{B} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta) = -1$$

we obtain readily that

$$\theta = -\frac{\pi}{4}$$

From above:

$$A = \frac{1}{2\cos\theta} = \frac{1}{2\cos\left(-\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}$$

From these results, the particular solution is

$$V_p(t) = \mathrm{A}\cos(\mathrm{t}+\theta) = \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$$

and the complete solution is

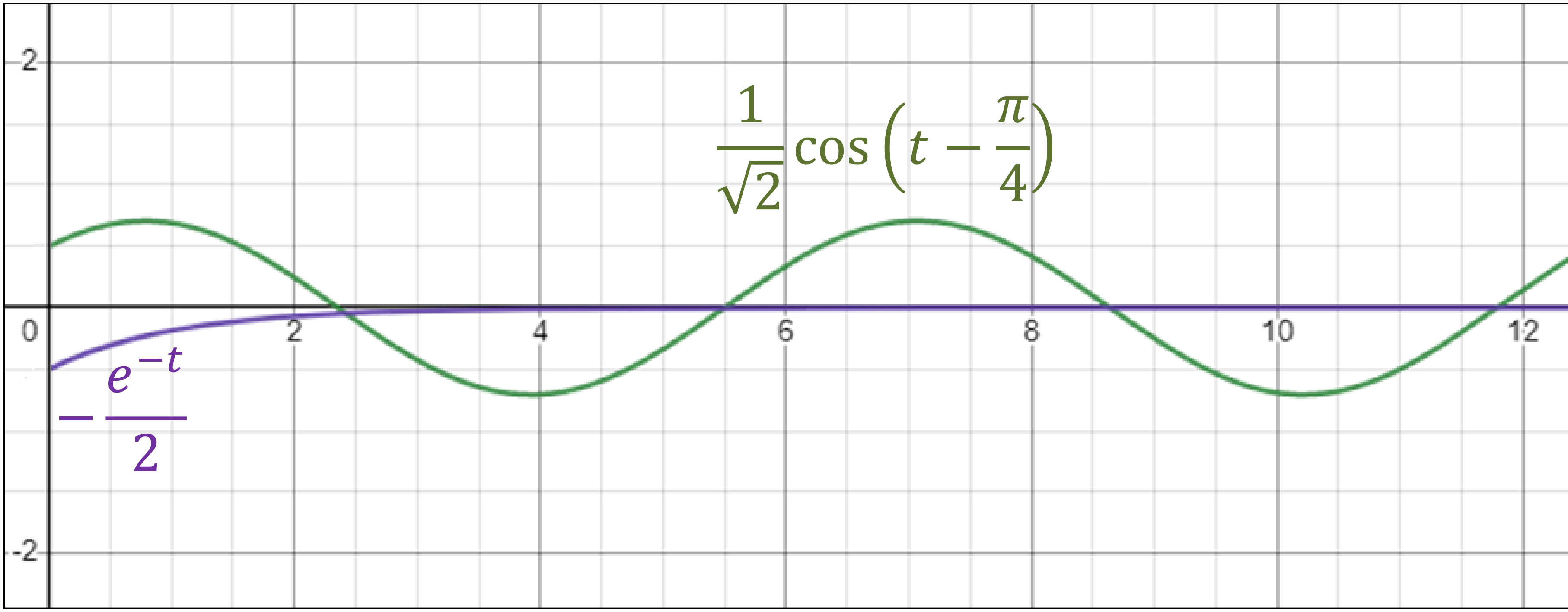
$$V_C(t) = K_1 e^{-t} + \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$$

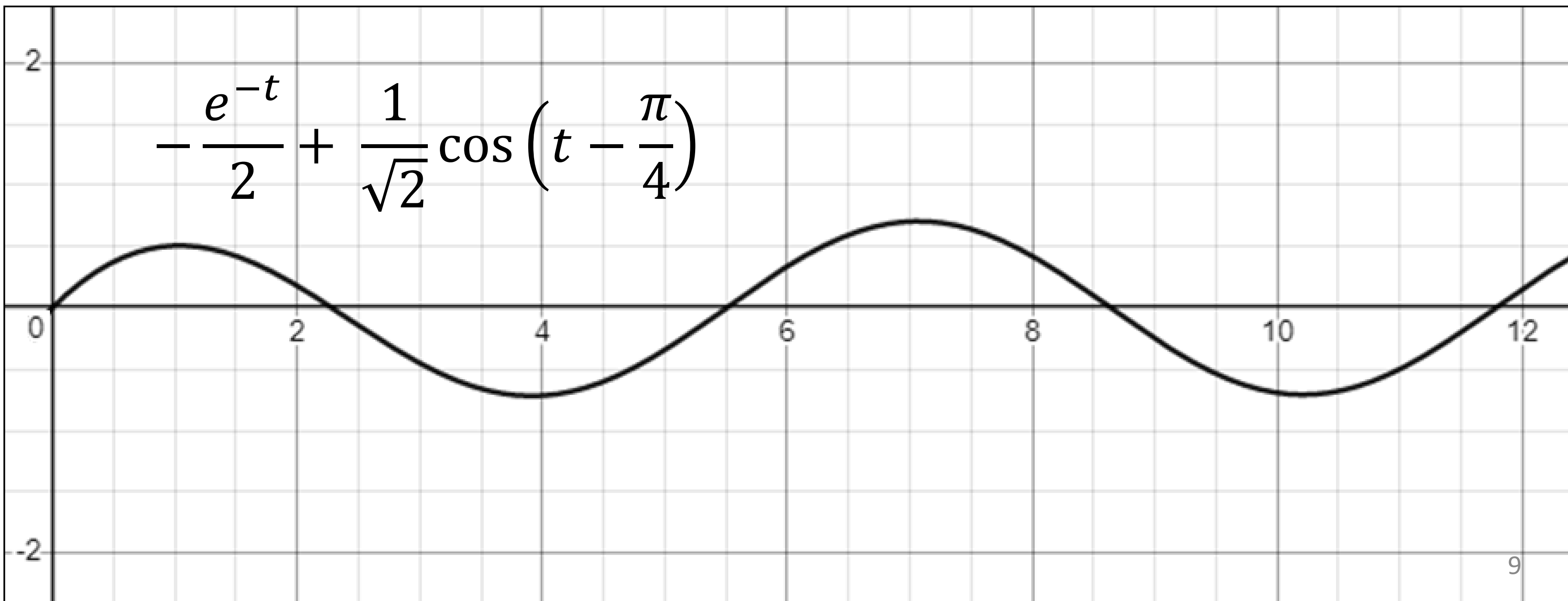
At $t = 0^+$

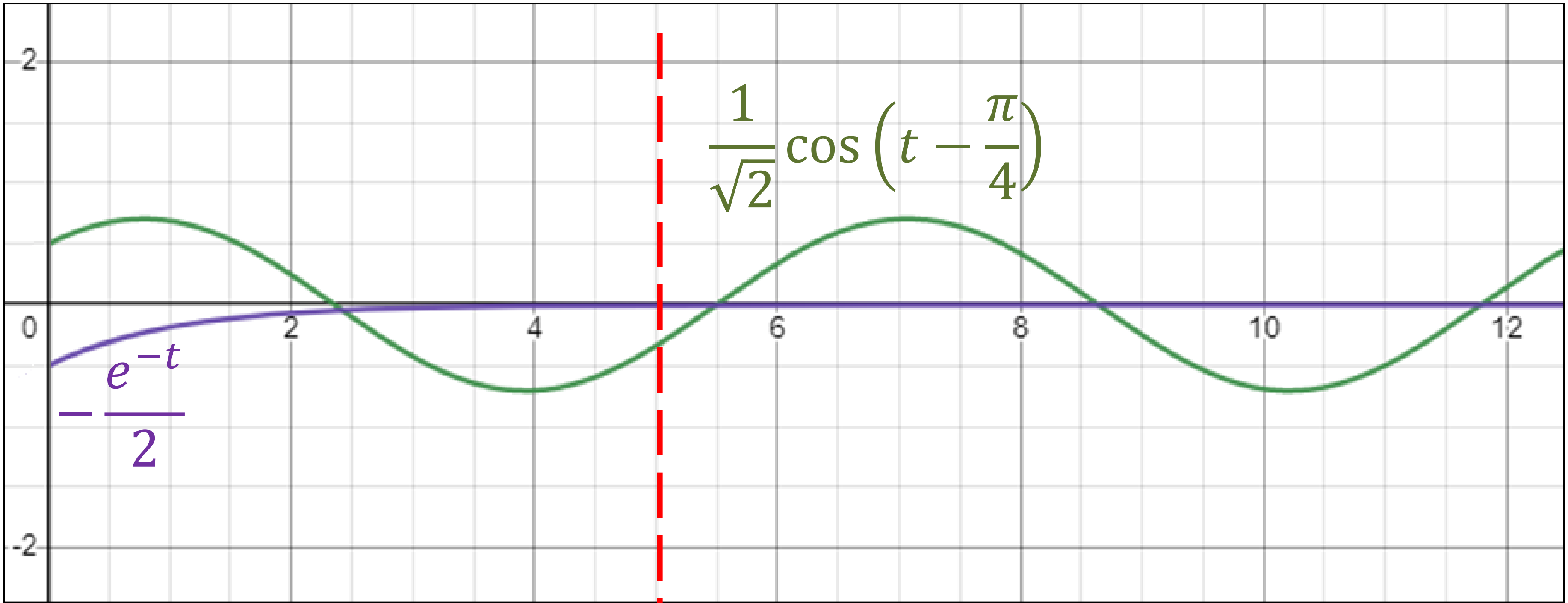
$$0 = K_1 + \frac{1}{\sqrt{2}}\cos\left(-\frac{\pi}{4}\right) = K_1 + \frac{1}{2} \Rightarrow \boxed{K_1 = -\frac{1}{2}}$$

Finally:

$$\boxed{V_C(t) = -\frac{e^{-t}}{2} + \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)}$$


$\frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$
$-\frac{e^{-t}}{2}$
2
-2
0
2
4
6
8
10
12


$-\frac{e^{-t}}{2}+\frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$
2
-2
0
2
4
6
8
10
12



Transient ⟵ ⟶ **≈ Steady-state**

In many practical situations the transients are very short. We are interested in describing steady-state system response with a more immediate mathematical approach for a specified frequency of operation. This will be accomplished next by introducing the phasor formalism.

Brief review of numbers

1) **Natural numbers** (even & odd #, prime #)

   1, 2, 3, 4, … → [∞ (infinity excluded)

2) **Whole numbers**

   0, 1, 2, 3, 4, … → [∞

3) **Integers**

   -∞] ← … , -4, -3, -2, -1, 0, 1, 2, 3, 4, … → [∞

4) **Rational numbers:** result from ratios of integers a/b with $b \neq 0$

5) **Irrational numbers:** All numbers which cannot be expressed by a ratio (examples: $\sqrt[n]{\text{prime \#}}, \pi, e$)

6) **Real numbers:** include all of the above

7) **Imaginary numbers:** Real # multiplied by $\sqrt{-1} = i$

8) **Complex numbers:** pair of (Real, Imaginary)

Imaginary numbers

For a long time it was believed that Real numbers satisfied all mathematical needs. However, it was observed that a Real number could not be the solution to an equation like (with a real)

$$x^2 + a = 0 \Rightarrow x = \pm\sqrt{a} \cdot \sqrt{-1}$$

Similar situations were encountered in a growing number of algebraic problems. To handle the square root $\sqrt{-1}$ a new type of number was introduced, which was called "imaginary" by defining

$$\sqrt{-1} = i$$

Complex numbers

Eventually, Complex numbers were formulated as an extension of the ordinary x-y plane, to form a number space representing all algebraic solutions.

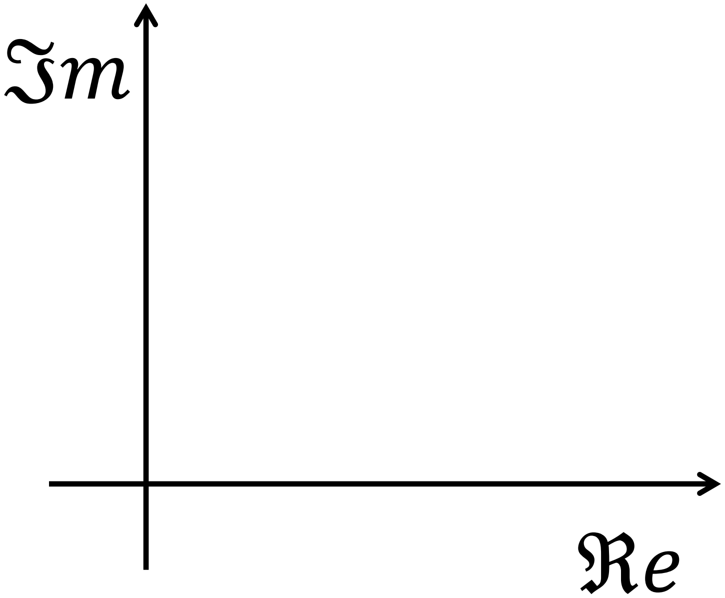
"x" represents all real numbers and "y" all imaginary numbers (real numbers multiplied by $\sqrt{-1} = i$).

Since in electrical engineering we use typically

$$i = \text{current}$$

it is customary to define instead

$$\sqrt{-1} = \mathrm{j}$$



Complex numbers are defined by points or vectors in the complex plane and can be represented in Cartesian coordinates

$$z = a + \mathrm{j}b$$

or in polar form

$$z = r\exp(\mathrm{j}\,\theta) = r\cos(\theta) + \mathrm{j}\,r\sin(\theta)$$

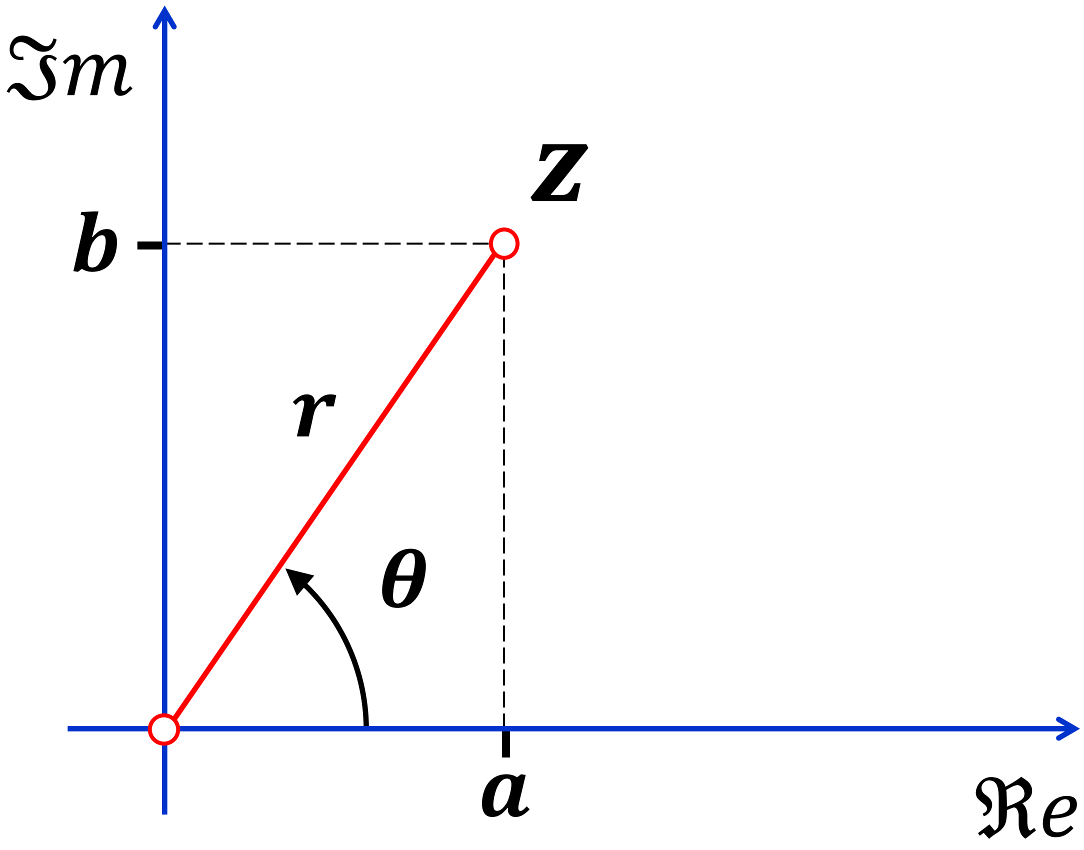
with $a = r\cos(x) = \text{real part}$

$b = r\sin(x) = \text{imaginary part}$

where

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{magnitude}$$

$$\theta = \angle z = \tan^{-1}\left(\frac{b}{a}\right) \quad \text{phase}$$



Note that:

$$z = r\,\exp(\,\mathrm{j}\theta) = r\,\exp(\,\mathrm{j}\theta \pm \mathrm{j}2n\pi)$$

Example 1:

Express $z_1 = 1 + \mathrm{j}1$ in polar form.

Amplitude

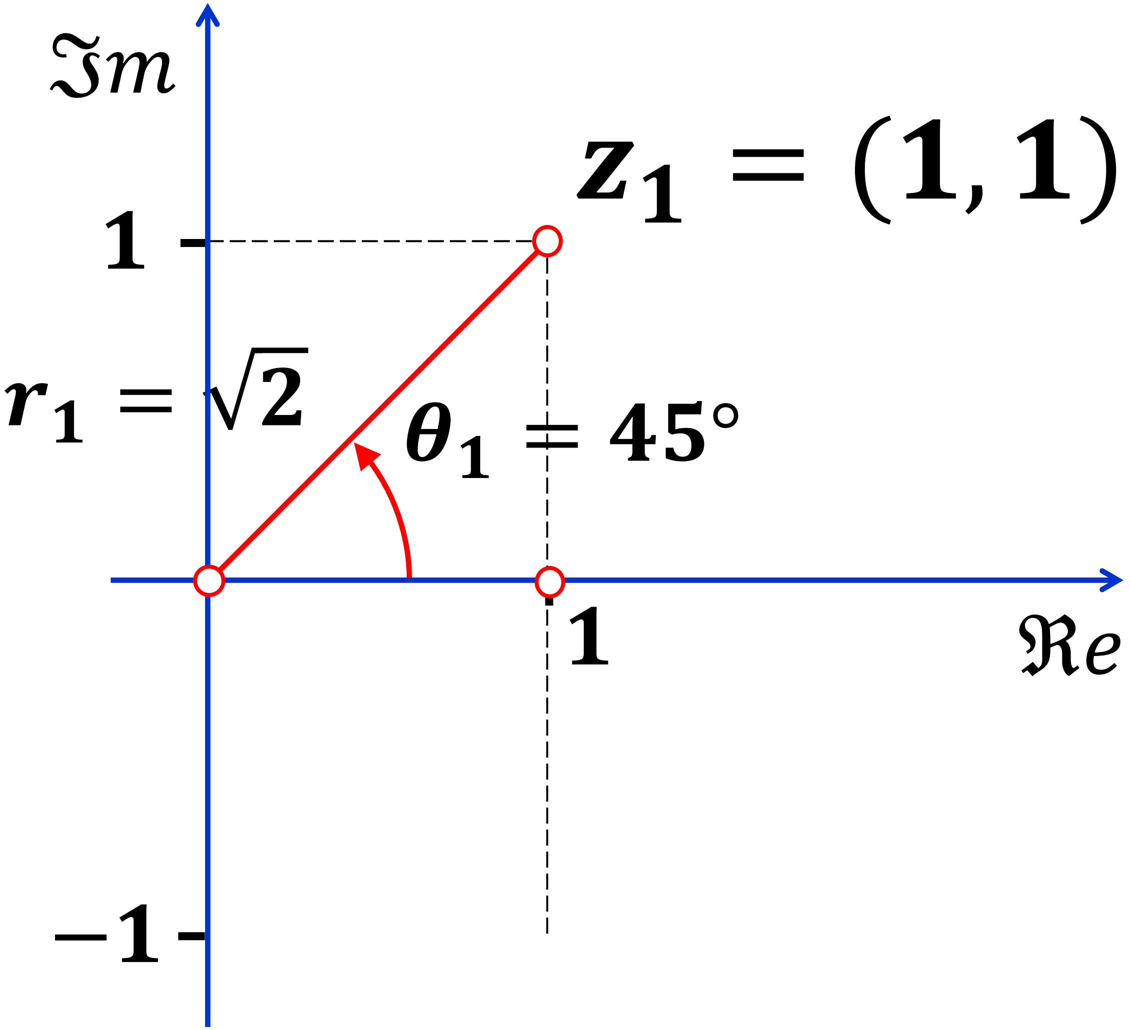
$$r_1 = |z_1| = \sqrt{a_1^2 + b_1^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Phase

$$\theta_1 = \angle z_1 = \tan^{-1}\left(\frac{b_1}{a_1}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} = 45°$$

Polar form

$$z_1 = \sqrt{2}\, e^{\,\mathrm{j}\frac{\pi}{4}}$$


ℑm
$z_1 = (1, 1)$
1
$r_1 = \sqrt{2}$
$\theta_1 = 45°$
1
ℜe
−1

Example 2:

Express $z_2 = 1 - \mathrm{j}1$ **in polar form.**

Amplitude

$$r_2 = |z_2| = \sqrt{a_2^2 + b_2^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

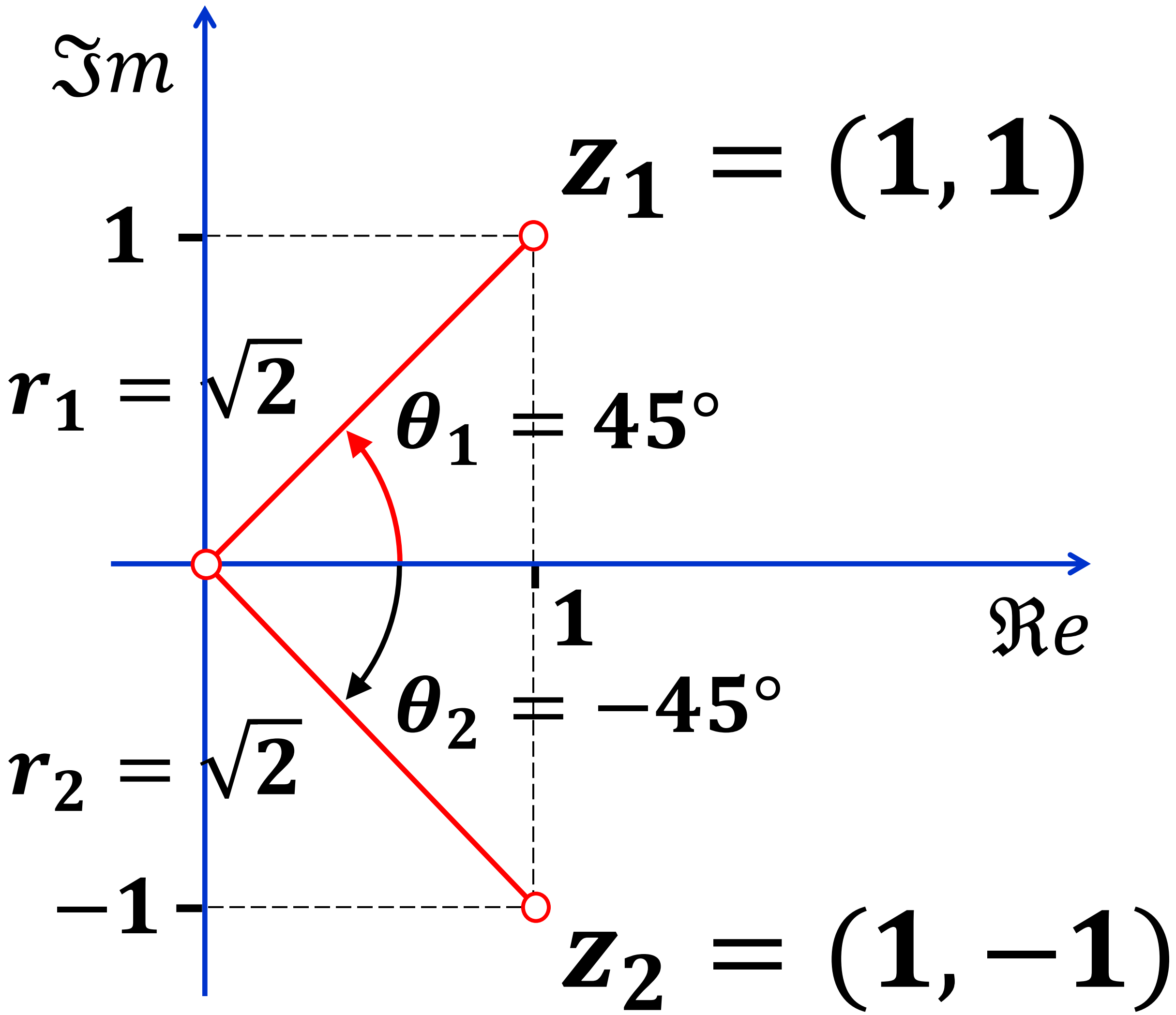
Phase

$$\theta_2 = \angle\, z_2 = \tan^{-1}\left(\frac{b_2}{a_2}\right) = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

$$= -45^\circ$$

Polar form

$$z_2 = \sqrt{2}\, e^{-\mathrm{j}\frac{\pi}{4}}$$



$\Im m$
$z_1 = (1, 1)$
1
$r_1 = \sqrt{2}$
$\theta_1 = 45°$
1
$\Re e$
$\theta_2 = -45°$
$r_2 = \sqrt{2}$
−1
$z_2 = (1, -1)$

Example 3:

Express $z_3 = -1 + \mathrm{j}1$ in polar form.

Amplitude

$$r_3 = |z_3| = \sqrt{a_3^2 + b_3^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

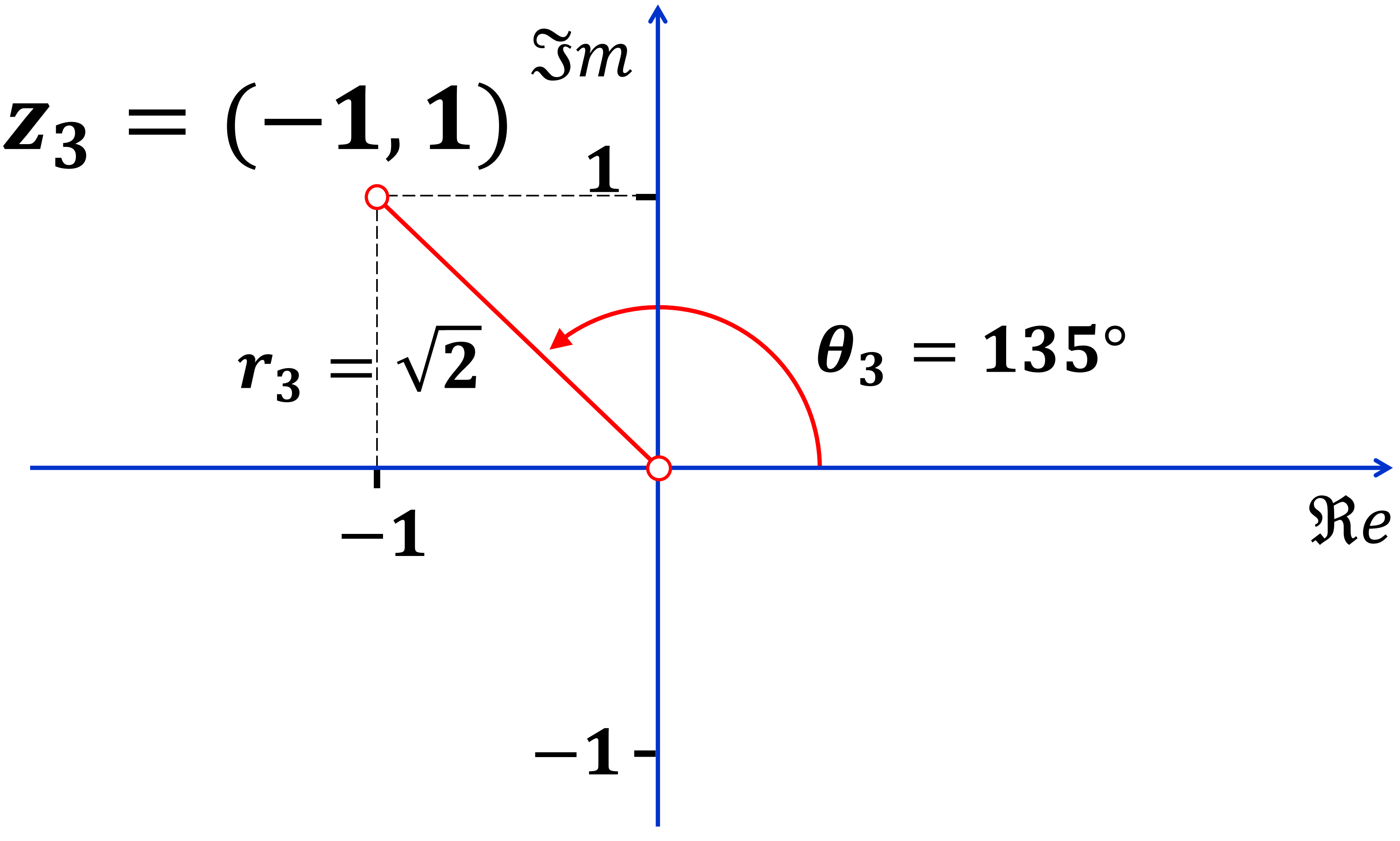
Phase

$$\theta_3 = \angle z_3 = \tan^{-1}\left(\frac{b_3}{a_3}\right) = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4}$$

$$= 135^\circ$$

Polar form

$$z_3 = \sqrt{2}\, e^{\mathrm{j}\frac{3\pi}{4}}$$


$z_3 = (-1, 1)$
$\Im m$
1
$r_3 = \sqrt{2}$
$\theta_3 = 135°$
-1
$\Re e$
-1

Every complex number has a complex conjugate

$$z^* = (a + \mathrm{j}b)^* = a - jb$$

such that

$$z \cdot z^* = (a + \mathrm{j}b) \cdot (a - \mathrm{j}b)$$

$$= a^2 + \mathrm{j}ab - \mathrm{j}ab + (\,\mathrm{j}b)(-\mathrm{j}b)$$

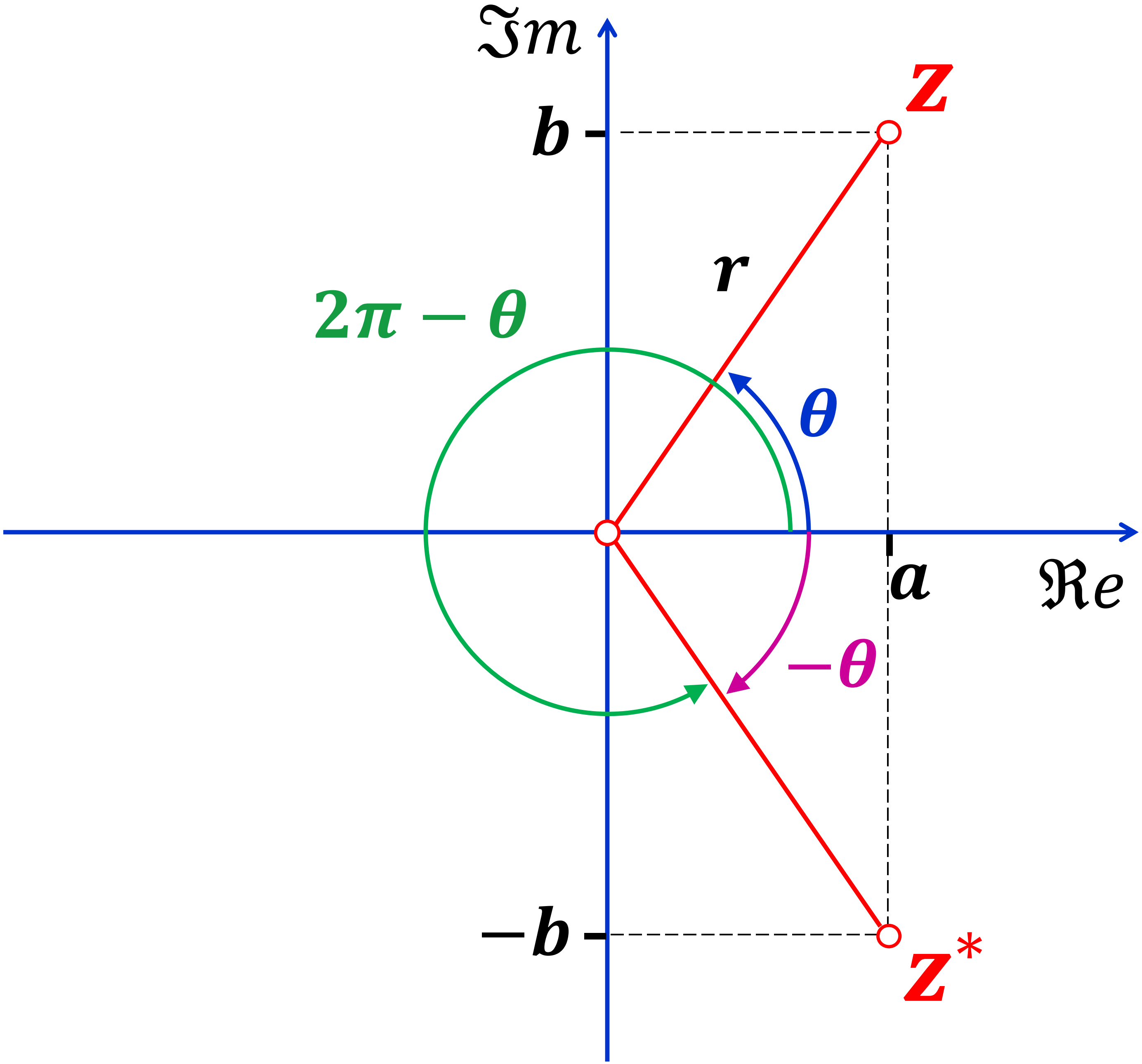
$$= a^2 + b^2 = |z|^2 = r^2$$

In polar form

$$z^* = [r \exp(\,\mathrm{j}\theta)]^*$$

$$= r \exp(\,-\mathrm{j}\theta) = r \exp(\,\mathrm{j}\,2\pi - \mathrm{j}\,\theta)$$

$$= r \cos(\theta) - \mathrm{j}\,r \sin(\theta)$$



$\Im m$
z
b
r
$2\pi - \theta$
θ
a
$\Re e$
$-\theta$
$-b$
z^*

The polar form is more useful in some cases.

For the power of a complex number

$$\boldsymbol{z^n = (a + \mathrm{j}b)^n = (a + \mathrm{j}b)(a + \mathrm{j}b)\cdots(a + \mathrm{j}b)}$$

the Cartesian form is quite cumbersome.

In polar form the result is immediate

$$\boldsymbol{z^n = [r\,\exp(\,\mathrm{j}\theta)]^n = r^n \exp(\,\mathrm{j}\,n\,\theta)}$$

Also notice that for multiples of 2π:

$$\boldsymbol{\exp(\,\mathrm{j}\,2n\pi\,) = 1}$$

In the case of roots, use the phase $\theta + 2k\pi$ (with k an integer) not to miss any results (there are n roots):

$$\sqrt[n]{z} = \sqrt[n]{r\ \exp(\,\mathrm{j}\theta + \mathrm{j}2k\pi)}$$

$$= \sqrt[n]{r}\ \exp\left[j\,\frac{\theta}{n} + j\,\frac{2k\pi}{n}\right]$$

$$0 \leq \frac{\theta}{n} \leq 2\pi$$

In engineering problems, the following identities are often useful for mathematical manipulations:

$$\mathbf{j} = \exp\left(\mathbf{j}\frac{\pi}{2}\right) \qquad -\mathbf{j} = \exp\left(-\mathbf{j}\frac{\pi}{2}\right)$$

The relations linking exponentials to trigonometric functions of complex variables are also widely used:

$$\cos(z) = \frac{\exp(\mathbf{j}\,z) + \exp(-\mathbf{j}\,z)}{2}$$

$$\sin(z) = \frac{\exp(\mathbf{j}\,z) - \exp(-\mathbf{j}\,z)}{2\mathbf{j}}$$

These result from Euler's identities

$$\exp(\pm\,\mathbf{j}\,z) = \cos(z) \pm \mathbf{j}\sin(z)$$

Examples:

1) Express in polar form $\boldsymbol{z = 1}$

Amplitude $\boldsymbol{r = |z| = 1}$

Phase $\boldsymbol{\theta = \angle z = 0^\circ}$

Polar form $\boldsymbol{z = 1\, e^{\,j\,0}}$

Examples:

2) Express in polar form $\boldsymbol{z = \mathrm{j}1}$

Amplitude $\boldsymbol{r = |z| = 1}$

Phase $\boldsymbol{\theta = \angle z = \frac{\pi}{2} = 90^{\circ}}$

Polar form $\boldsymbol{z = 1\, e^{\,\mathrm{j}\frac{\pi}{2}}}$

Examples:

3) Express in polar form $z = -1$

Amplitude $r = |z| = 1$

Phase $\theta = \angle z = \pi = 180°$

Polar form $z = 1\, e^{\,\mathrm{j}\,\pi}$

Examples:

4) Express in polar form $$z = \frac{1 - j1}{1 + j1}$$

Amplitude $$r = |z| = \frac{|1 - j1|}{|1 + j1|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

Phase $$\theta = \angle z = \angle(1 - j1) - \angle(1 + j1) =$$
$$= \left(-\frac{\pi}{4}\right) - \left(\frac{\pi}{4}\right) = -\frac{\pi}{2}$$

Polar form $$z = 1\, e^{-j\frac{\pi}{2}}$$

Ratio of complex numbers

$$z = \frac{a + \mathrm{j}b}{c + \mathrm{j}d} = \frac{r_1 e^{\mathrm{j}\theta_1}}{r_2 e^{\mathrm{j}\theta_2}} = \frac{r_1}{r_2} e^{\mathrm{j}(\theta_1 - \theta_2)}$$

$$|z| = \frac{|z_1|}{|z_2|} \qquad\qquad \angle z = \angle z_1 - \angle z_2$$

You can also rationalize to make denominator real

$$z = \frac{a + \mathrm{j}b}{c + \mathrm{j}d} = \frac{(a + \mathrm{j}b)(c - \mathrm{j}d)}{(c + \mathrm{j}d)(c - \mathrm{j}d)}$$

$$= \frac{ac + bd + \mathrm{j}(bc - ad)}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \mathrm{j}\frac{(bc - ad)}{c^2 + d^2}$$

Going back to example 4)

$$z = \frac{1 - j1}{1 + j1} = \frac{1 - j1}{1 + j1} \times \underbrace{\frac{1 - j1}{1 - j1}}_{} = 1$$

complex conjugate $(1 + j1)(1 - j1) = 2$

$$z = \frac{1 - j2 + \overbrace{(j^2)}^{= -1}}{2} = \frac{1 - 1 - j2}{2} = -j$$

Amplitude $r = 1$

Phase $\theta = -\frac{\pi}{2}$

Polar form $z = 1\, e^{-j\frac{\pi}{2}}$

Examples:

5) Express in Cartesian (rectangular) form: $z = 3\, e^{-j\frac{\pi}{6}}$

$$z = 3\, e^{-j\frac{\pi}{6}} = 3\left[\cos\left(-\frac{\pi}{6}\right) + j\,\sin\left(-\frac{\pi}{6}\right)\right]$$

$$z = 3\left[\frac{\sqrt{3}}{2} + j\left(-\frac{1}{2}\right)\right]$$

$$z = \frac{3\sqrt{3}}{2} - j\,\frac{3}{2}$$

Examples:

6) Express in Cartesian (rectangular) form:

$$z = 3\,e^{j\frac{\pi}{6}} + 3\,e^{-j\frac{\pi}{6}}$$

$$z = 3\left[\cos\left(\frac{\pi}{6}\right) + j\sin\left(\frac{\pi}{6}\right)\right] + 3\left[\cos\left(-\frac{\pi}{6}\right) + j\sin\left(-\frac{\pi}{6}\right)\right]$$

$$z = 3\left[\frac{\sqrt{3}}{2} + j\left(\frac{1}{2}\right)\right] + 3\left[\frac{\sqrt{3}}{2} + j\left(-\frac{1}{2}\right)\right]$$

$$z = \frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} + j\frac{3}{3} - j\frac{3}{2} = 3\sqrt{3} - j0$$

Examples:

7) Express in polar form

$$z = \frac{(-1+\mathrm{j})^5}{1+\mathrm{j}}$$

As found earlier

$$-1+\mathrm{j} = \sqrt{2}\, e^{\mathrm{j}\frac{3\pi}{4}} \qquad 1+\mathrm{j} = \sqrt{2}\, e^{\mathrm{j}\frac{\pi}{4}}$$

$$(-1+\mathrm{j})^5 = \left(\sqrt{2}\right)^5 e^{\mathrm{j}\,5\,\frac{3\pi}{4}} = 4\sqrt{2}e^{\mathrm{j}\frac{15\pi}{4}}$$

$$z = \frac{4\sqrt{2}e^{\mathrm{j}\frac{15\pi}{4}}}{\sqrt{2}\, e^{\mathrm{j}\frac{\pi}{4}}} = 4e^{\mathrm{j}\frac{14\pi}{4}} = 4e^{\mathrm{j}\frac{7\pi}{2}}$$

$$z = 4e^{\mathrm{j}\left(2\pi+\frac{3\pi}{2}\right)} = 4\, \underbrace{e^{\mathrm{j}2\pi}}_{=1}\, e^{\mathrm{j}\frac{3\pi}{2}} = 4e^{\mathrm{j}\frac{3\pi}{2}}$$

When time-harmonic functions are considered, it is possible to simplify the analysis of engineering systems by using complex representation.

Example of time-harmonic function:

$$A\cos(\omega t + \theta)$$

Amplitude

Angular frequency
$\omega = 2\pi f$

phase

By invoking Euler's identity, we can write

$$A\cos(\omega t + \theta) =$$

$$= \Re e[A\cos(\omega t + \theta) + \mathrm{j}\, A\sin(\omega t + \theta)]$$

$$= \Re e[A\exp(\mathrm{j}\omega t + \mathrm{j}\theta)]$$

Now we are going to use the properties of the exponentials to split frequency from phase:

$$\Re e[A\exp(\,\mathrm{j}\omega t+\mathrm{j}\theta)\,]=$$

$$\Re e[A\exp(\,\mathrm{j}\omega t)\exp(\,\mathrm{j}\theta)]=$$

$$\Re e[\underbrace{A\exp(\,\mathrm{j}\theta)}_{\text{phasor of the time-harmonic function}}\exp(\,\mathrm{j}\omega t)]$$

The "phasor" contains the essential information on amplitude and phase.

For a known frequency ω, $A\exp(j\theta)$ characterizes completely $A\cos(\omega t+\theta)$.