ECE 205 “Electrical and Electronics Circuits”

Spring 2024 – LECTURE 19

MWF – 12:00pm

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2062 ECE Building

Lecture 19 – Summary

Learning Objectives

1. Phasor representation of circuit problems in sinusoidal regime

Suppose you need to add two time-harmonic functions

$$v_1(t) = A_1 \cos(\omega t + \theta_1)$$

$$v_2(t) = A_2 \cos(\omega t + \theta_2)$$

With trigonometry you have to use cumbersome formulas like:

$$\cos \mathrm{X} + \cos \mathrm{Y} = 2 \cos\left(\frac{X+Y}{2}\right) \cos\left(\frac{X-Y}{2}\right)$$

In phasor form:

$$v_1(t) = A_1 \cos(\omega t + \theta_1) = \Re e[A_1 \exp(\mathrm{j}\omega t + \mathrm{j}\theta_1)]$$

$$\Leftrightarrow \quad \mathrm{V}_1 = A_1 \exp(\mathrm{j}\theta_1) = A_1 \angle \theta_1$$

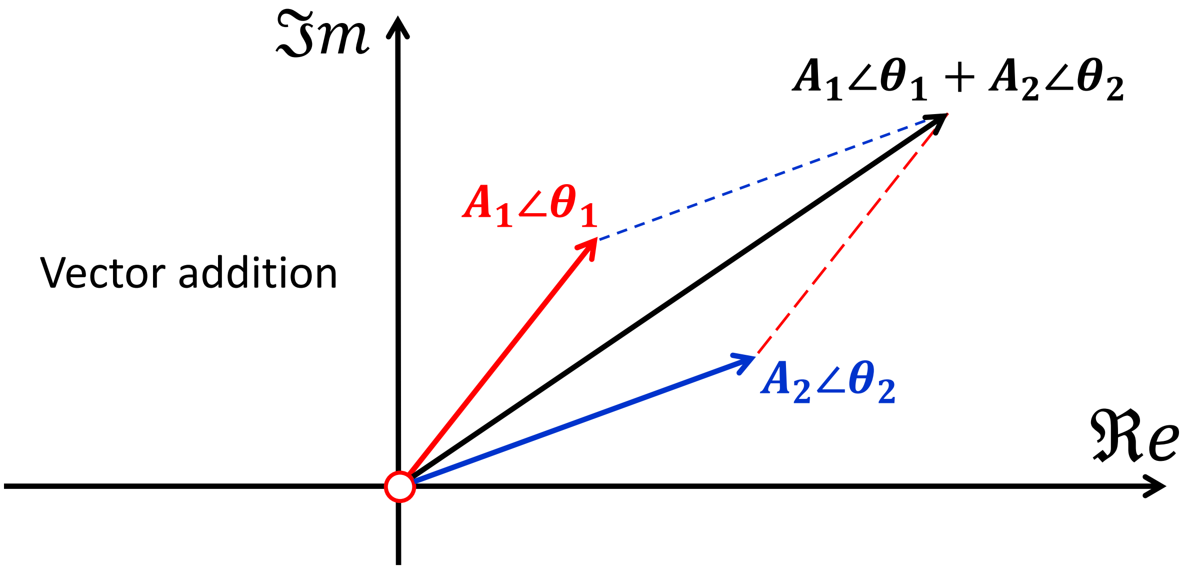
$$v_2(t) = A_2 \cos(\omega t + \theta_2) = \Re e[A_2 \exp(\mathrm{j}\omega t + \mathrm{j}\theta_2)]$$

$$\Leftrightarrow \quad \mathrm{V}_2 = A_2 \exp(\mathrm{j}\theta_2) = A_2 \angle \theta_2$$

$$v_1(t) + v_2(t) \Leftrightarrow \mathrm{V}_1 + \mathrm{V}_2 = A_1 \exp(\mathrm{j}\theta_1) + A_2 \exp(\mathrm{j}\theta_2)$$

CAUTION: ⇔ is a "transformation" NOT an "equality"!

Vector addition



Example – Express the following in its phasor form:

$$v(t) = 2\sqrt{2}\,\sin\left(1000t + \frac{\pi}{4}\right) + 2\sqrt{2}\,\cos\left(1000t + \frac{\pi}{4}\right)$$

First term:

$$v_1(t) = 2\sqrt{2}\,\cos\left(1000t + \frac{\pi}{4} - \frac{\pi}{2}\right)$$

$$v_1(t) = 2\sqrt{2}\,\cos\left(1000t - \frac{\pi}{4}\right)$$

$$\mathbf{V}_1 = 2\sqrt{2}\angle -\frac{\pi}{4} = 2\sqrt{2}\,e^{-j\frac{\pi}{4}}$$

$$= 2\sqrt{2}\left(\cos\frac{\pi}{4} - \text{j}\sin\frac{\pi}{4}\right)$$

Example – Express the following in its phasor form:

$$v(t) = 2\sqrt{2}\sin\left(1000t + \frac{\pi}{4}\right) + 2\sqrt{2}\cos\left(1000t + \frac{\pi}{4}\right)$$

Second term:

$$v_2(t) = 2\sqrt{2}\cos\left(1000t + \frac{\pi}{4}\right)$$

$$\mathbf{V}_2 = 2\sqrt{2}\angle\frac{\pi}{4} = 2\sqrt{2}\,e^{\mathrm{j}\frac{\pi}{4}}$$

$$= 2\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{j}\sin\frac{\pi}{4}\right)$$

Example – Express the following in its phasor form:

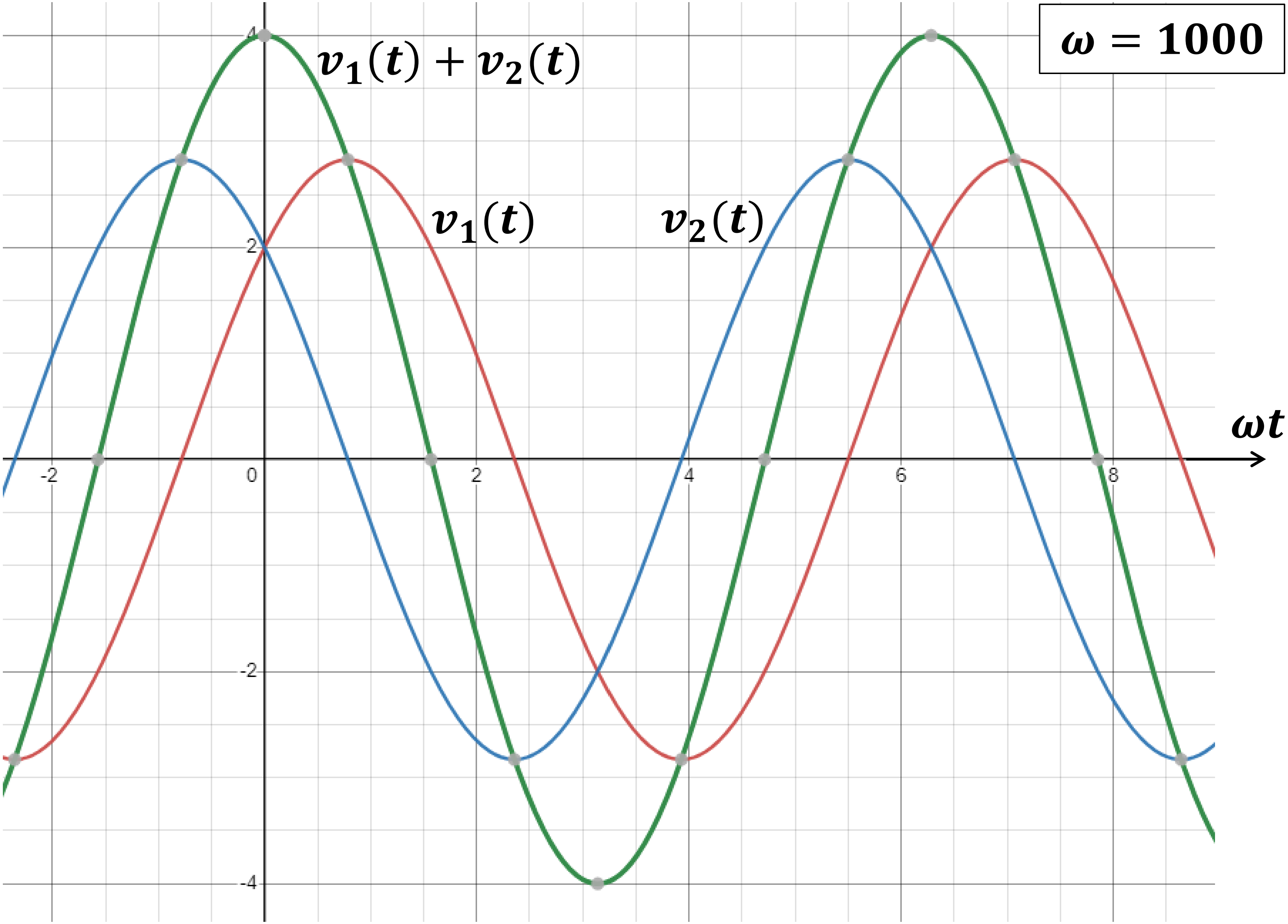
$$v(t) = 2\sqrt{2}\sin\left(1000t + \frac{\pi}{4}\right) + 2\sqrt{2}\cos\left(1000t + \frac{\pi}{4}\right)$$

Combine the results:

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = 2\sqrt{2}\left(\cos\frac{\pi}{4} - \text{j}\sin\frac{\pi}{4}\right) + 2\sqrt{2}\left(\cos\frac{\pi}{4} + \text{j}\sin\frac{\pi}{4}\right)$$

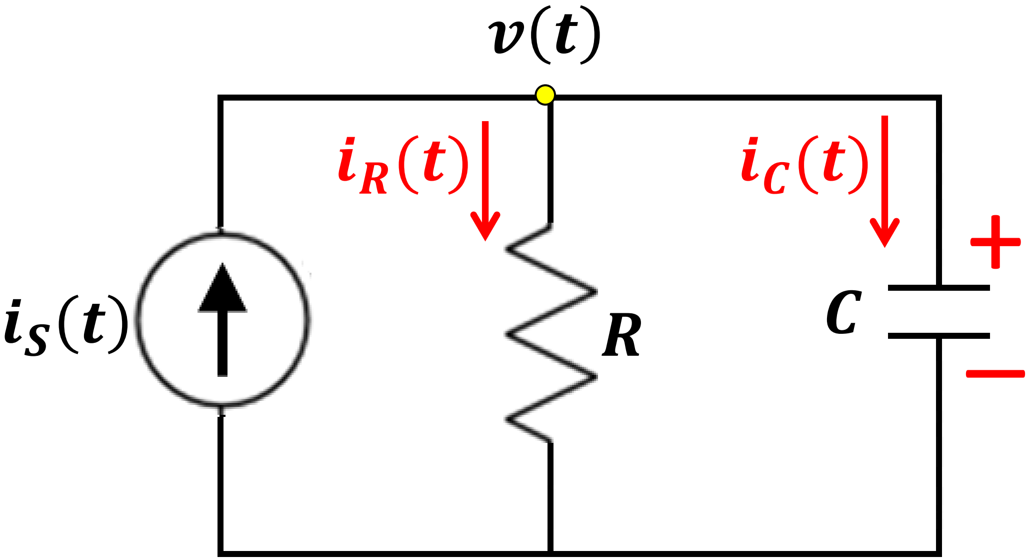
$$= 2\sqrt{2}\left(2\cos\frac{\pi}{4}\right) = 2\sqrt{2}\left(2\sqrt{2}/2\right) = 4\angle 0°$$

$$\Rightarrow \boxed{v(t) = 4\cos(1000t)\ [\text{V}]}$$



$\omega = 1000$
$v_1(t) + v_2(t)$
$v_1(t)$
$v_2(t)$
ωt
4
2
0
-2
-4
-2
2
4
6
8

RC Circuit Example with time-harmonic forcing term



Find $i(t)$ when $v(t)$ is measured as:

$$v(t) = \frac{10}{\sqrt{2}} \cos\left(1000t - \frac{\pi}{4}\right)$$

$R = 1\Omega$ $\quad$ $C = 1\text{mF}$

KCL

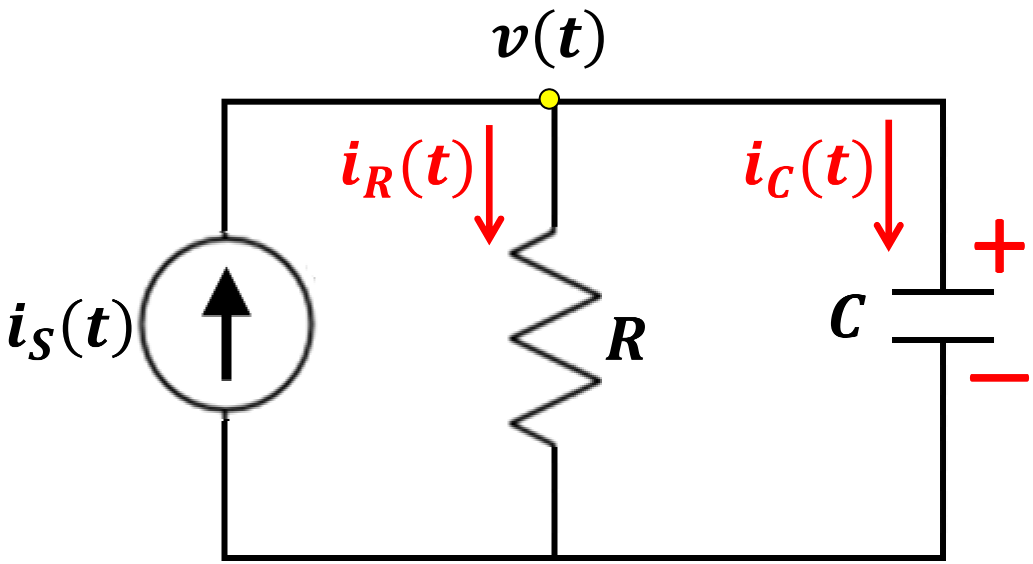
$$i_S = i_R(t) + i_C(t) = \frac{v(t)}{R} + C\frac{d\,v(t)}{dt}$$

$$= \frac{10}{1 \times \sqrt{2}} \cos\left(1000t - \frac{\pi}{4}\right) - 10^{-3}\frac{10^4}{\sqrt{2}} \sin\left(1000t - \frac{\pi}{4}\right)$$

$$= \frac{10}{\sqrt{2}} \cos\left(1000t - \frac{\pi}{4}\right) - \frac{10}{\sqrt{2}} \sin\left(1000t - \frac{\pi}{4}\right)$$

$$= \frac{10}{\sqrt{2}} \cos\left(1000t - \frac{\pi}{4}\right) + \frac{10}{\sqrt{2}} \cos\left(1000t + \frac{\pi}{4}\right)$$

RC Circuit Example with time-harmonic forcing term



Find $i(t)$ when $v(t)$ is measured as:

$$v(t) = \frac{10}{\sqrt{2}} \cos\left(1000t - \frac{\pi}{4}\right)$$

$R = 1\Omega$

$C = 1\text{mF}$

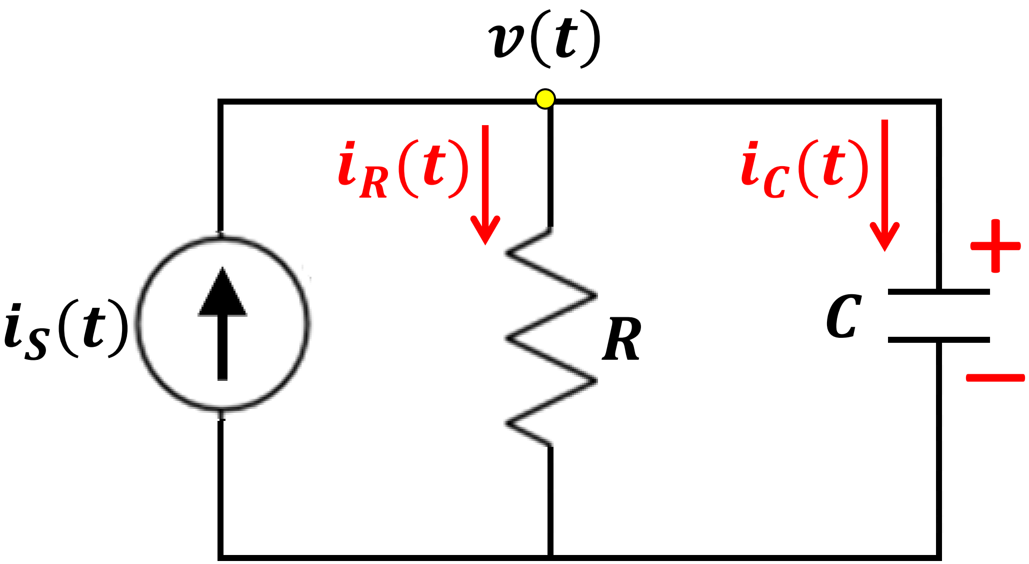
$$i_S(t) = \frac{10}{\sqrt{2}} \cos\left(1000t - \frac{\pi}{4}\right) + \frac{10}{\sqrt{2}} \cos\left(1000t + \frac{\pi}{4}\right)$$

Phasor form

$$\mathbf{I}_S = \frac{10}{\sqrt{2}} \angle -\frac{\pi}{4} + \frac{10}{\sqrt{2}} \angle \frac{\pi}{4}$$

$$= \frac{10}{\sqrt{2}} \exp\left(-j\frac{\pi}{4}\right) + \frac{10}{\sqrt{2}} \exp\left(j\frac{\pi}{4}\right)$$

RC Circuit Example with time-harmonic forcing term



Find $i(t)$ when $v(t)$ is measured as:

$$v(t) = \frac{10}{\sqrt{2}} \cos\left(1000t - \frac{\pi}{4}\right)$$

$R = 1\Omega$

$C = 1\text{mF}$

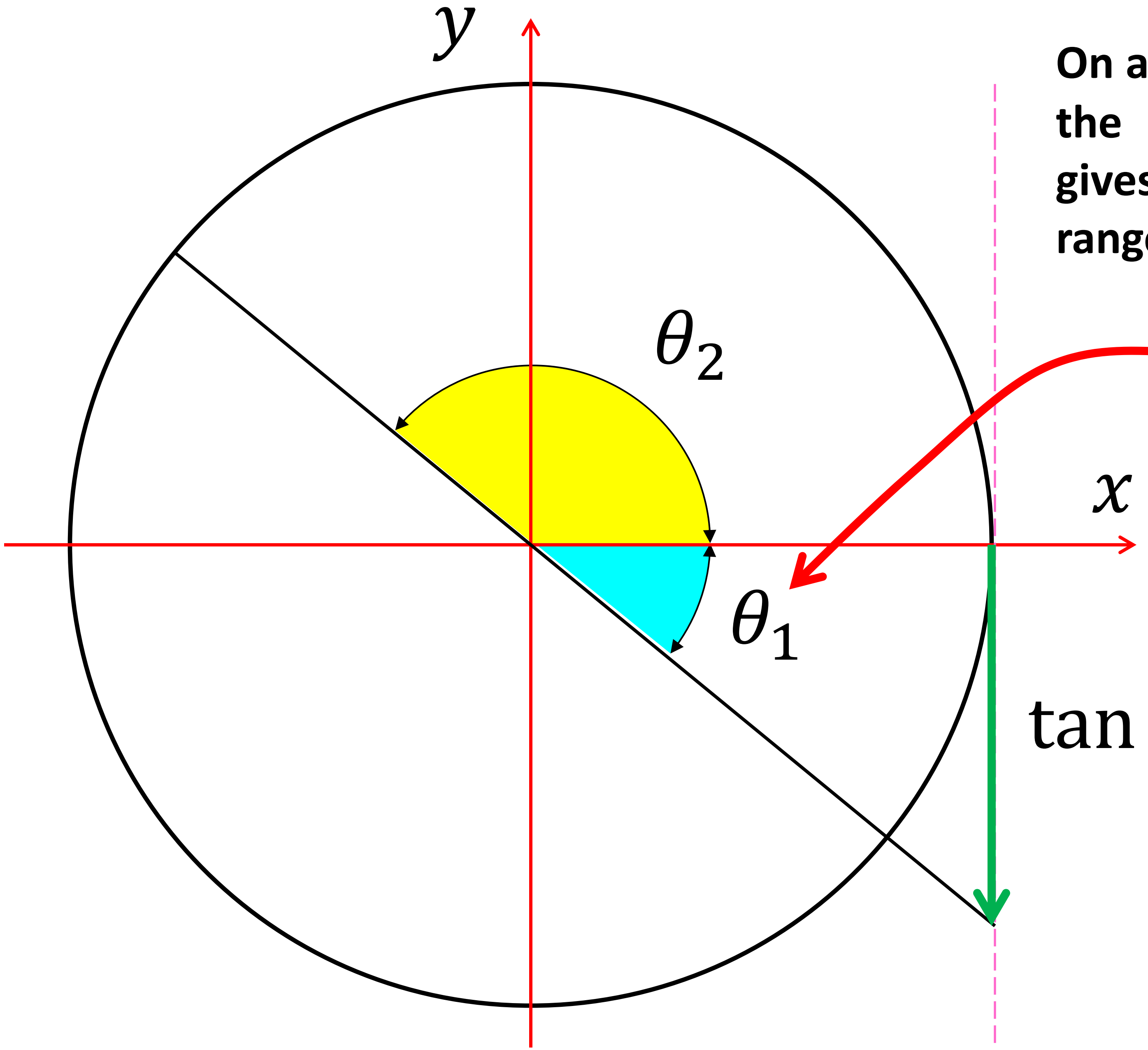
$$\mathrm{I}_S = \frac{10}{\sqrt{2}} \exp\left(-j\frac{\pi}{4}\right) + \frac{10}{\sqrt{2}} \exp(j\frac{\pi}{4})$$

$$= \frac{10}{\sqrt{2}}\left(\cos\frac{\pi}{4} - \mathrm{j}\sin\frac{\pi}{4} + \cos\frac{\pi}{4} + \mathrm{j}\sin\frac{\pi}{4}\right)$$

$$\mathrm{I}_S = \frac{10}{\sqrt{2}}\left(2\frac{\sqrt{2}}{2}\right) = 10\angle 0^\circ \Leftrightarrow \ i_S(t) = 10\cos(1000t)\ [\mathrm{A}]$$

NOTE: These two angles have the same tangent

On a typical calculator, the $\tan^{-1}$ operation gives an answer in the range



$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\tan\theta_1 = \tan\theta_2$$

Computer languages (and more advanced calculators) give the option of two different ranges for the arctangent operation

$$\mathrm{atan}(y/x)$$

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\mathrm{atan2}(y, x)$$

$$[-\pi, \pi]$$

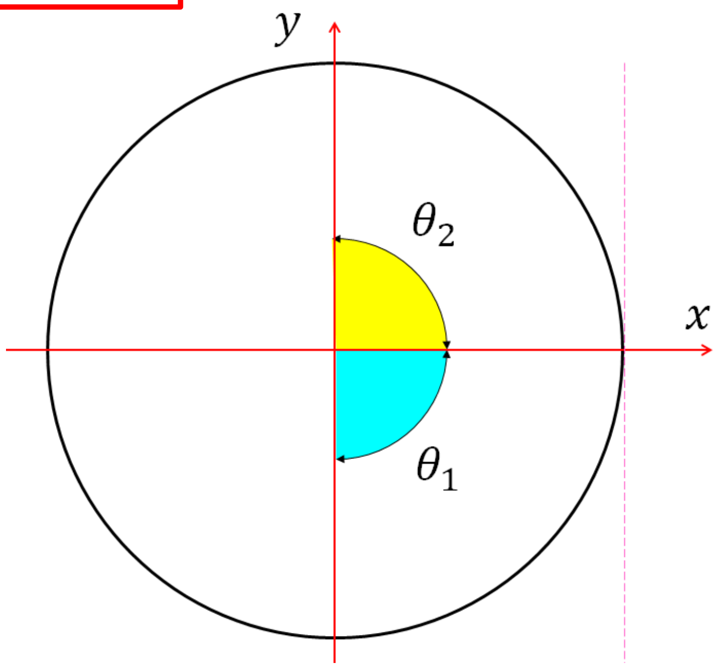
if $x = 0$

$y > 0$

$$\mathrm{atan2}(y) = \frac{\pi}{2}$$

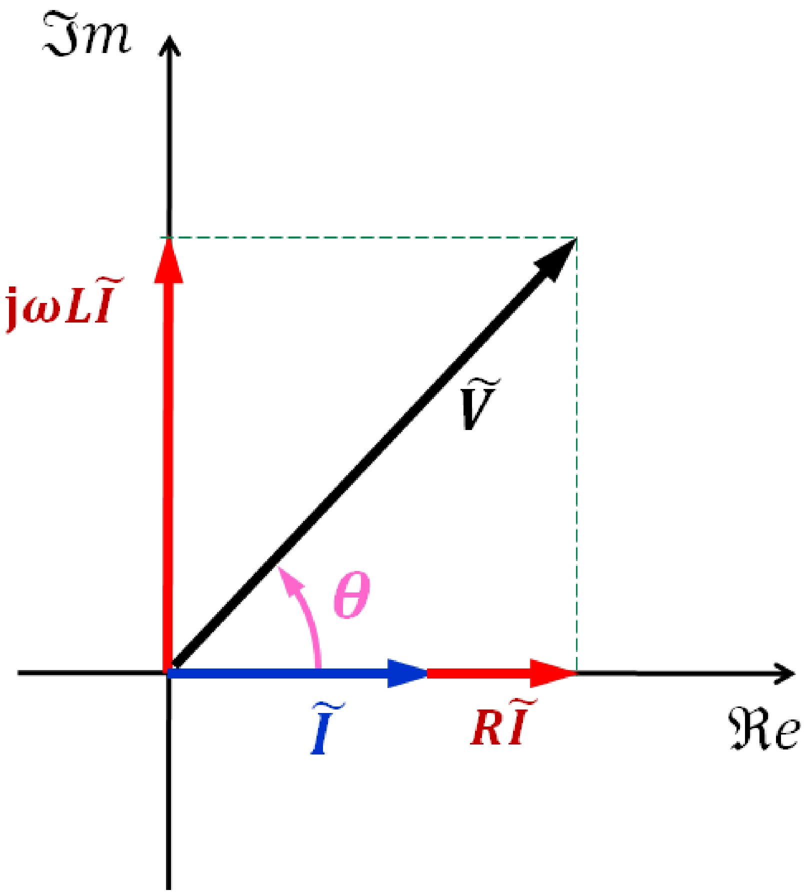
$y < 0$

$$\mathrm{atan2}(y) = -\frac{\pi}{2}$$



In circuits we consider the ratio between voltage and current phasors (called the impedance).

The real part of the impedance is the resistance, which is always positive in RLC circuits.



The angle between voltage and current in the complex plane is normally in the range

$$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

Time differentiation is greatly simplified with phasors.

Consider the time-dependent voltage

$$V(t) = V_0 \cos(\omega t + \theta)$$

with phasor

$$\tilde{V} = V_0 \exp(j\theta)$$

We wish to find the phasor representation for

$$\frac{dV(t)}{dt} = \frac{d}{dt} V_0 \cos(\omega t + \theta)$$

$$\frac{dV(t)}{dt} = \frac{d}{dt} V_0 \cos(\omega t + \theta)$$

Phasor Transform for the Derivative

$$\frac{dV(t)}{dt} \Leftrightarrow \mathrm{j}\omega V_0 \exp(\mathrm{j}\theta) = \mathrm{j}\omega \,\tilde{\mathrm{V}}$$

TIME DERIVATIVE

PHASOR TRANSFORM

PHASOR

PROOF

$$\frac{dV(t)}{dt} = \frac{d}{dt} V_0 \cos(\omega t + \theta)$$

$$= -\omega V_0 \sin(\omega t + \theta)$$

$$= -\omega\, V_0 \cos(\omega t + \theta - \pi/2\,)$$

$$= \Re e\{-\omega V_0 \exp(\,\mathrm{j}\omega t + \mathrm{j}\theta - \mathrm{j}\,\pi/2)\}$$

$$= \Re e\{-\omega V_0 \exp(\,\mathrm{j}\omega t) \exp(\,\mathrm{j}\theta) \exp(\,-\mathrm{j}\,\pi/2)\}$$

$$= \Re e\{-(-\mathrm{j})\omega V_0 \exp(\,\mathrm{j}\omega t) \exp(\,\mathrm{j}\theta)\}$$

$$= \Re e\{\,\mathrm{j}\omega\, V_0 \exp(\,\mathrm{j}\theta) \exp(\,\mathrm{j}\omega t)\}$$

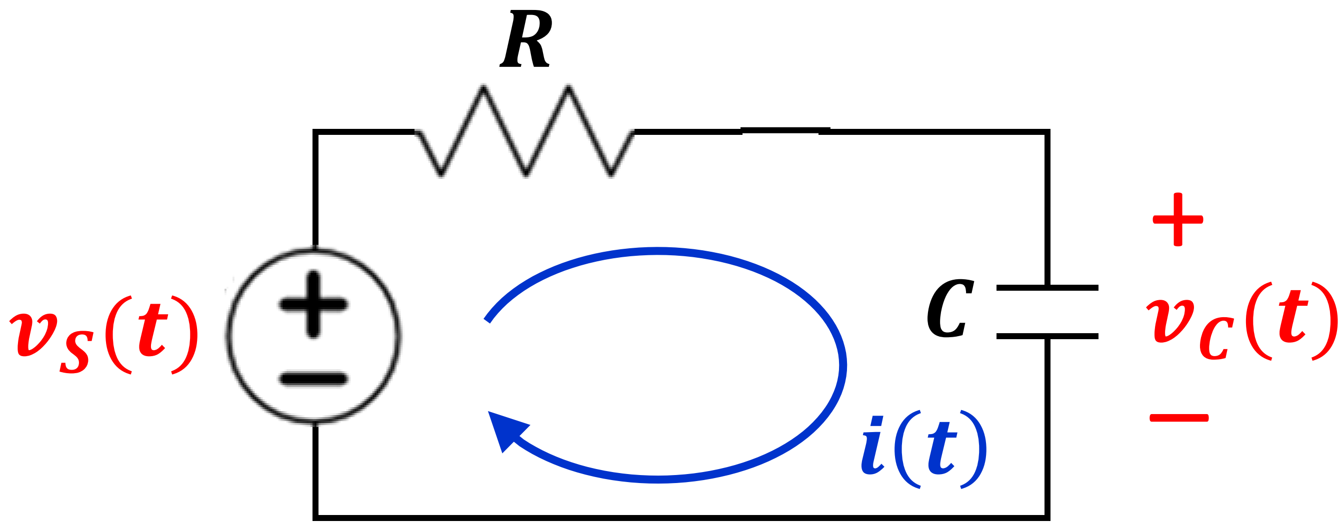
$$\frac{dV(t)}{dt} \Leftrightarrow \underbrace{\mathrm{j}\omega V_0 \exp(\,\mathrm{j}\theta)}\; = \mathrm{j}\omega\, \tilde{\mathrm{V}}$$

TIME DERIVATIVE

PHASOR TRANSFORM

PHASOR

Consider the RC circuit again



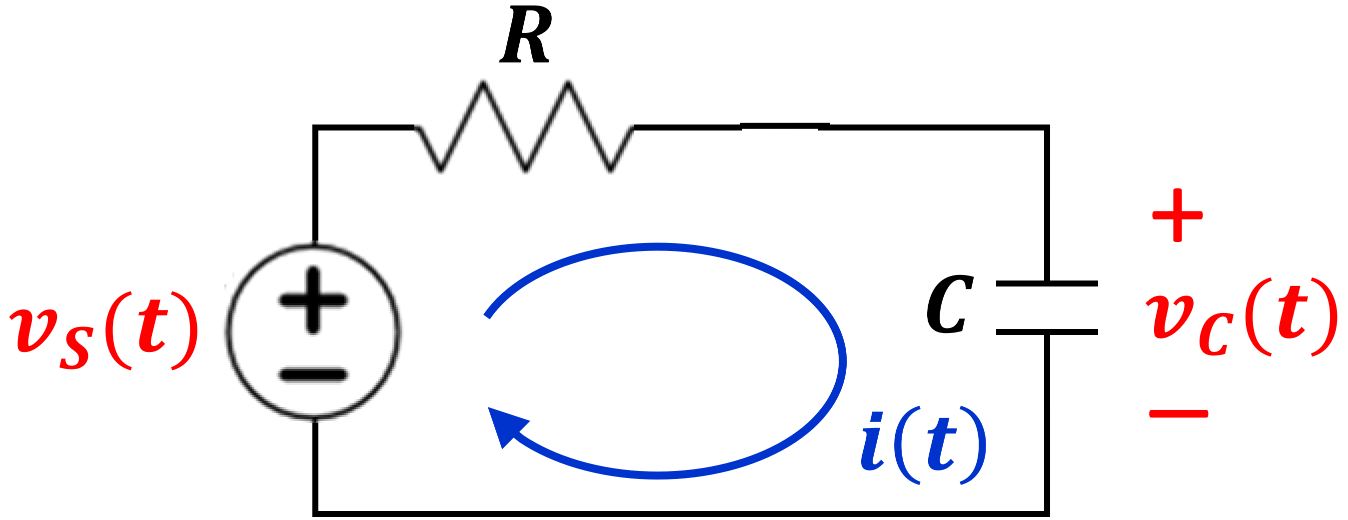
$$v_S(t) = V'_m \cos(\omega t + \theta)$$

The voltage across the capacitor has the form

$$v_c(t) = V_m \cos(\omega t + \theta_v)$$

(same frequency but magnitude and phase change)

Consider the RC circuit again



$$v_S(t) = V'_m \cos(\omega t + \theta)$$

Current $i(t)$ is given by

$$i(t) = C\frac{dv_C(t)}{dt} = C\,\frac{d}{dt}\,V_m \cos(\omega t + \theta_v)$$

$$v_c(t) = V_m\cos(\omega t + \theta_v)$$

$$i(t) = C\frac{dv_C(t)}{dt} = C\,\frac{d}{dt}\,V_m\cos(\omega t + \theta_v)$$

Voltage phasor across the capacitor

$$\tilde{V} \ = \ V_{\mathrm{m}}\angle\theta_v$$

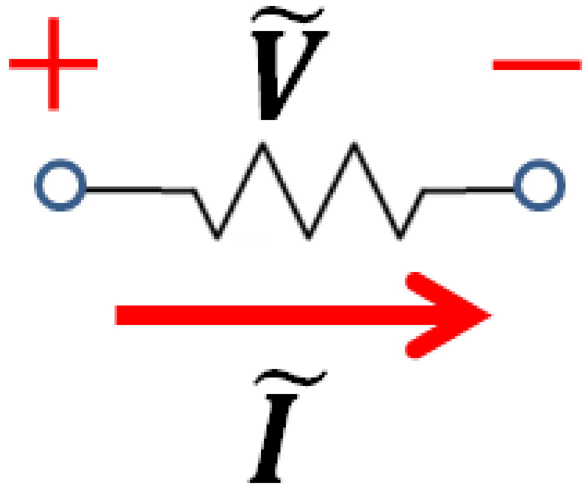
Current phasor across the capacitor

$$\tilde{I} \ = j\omega\, C\, V_{\mathrm{m}}\angle\theta_v$$

<u>Impedance</u> of the capacitor

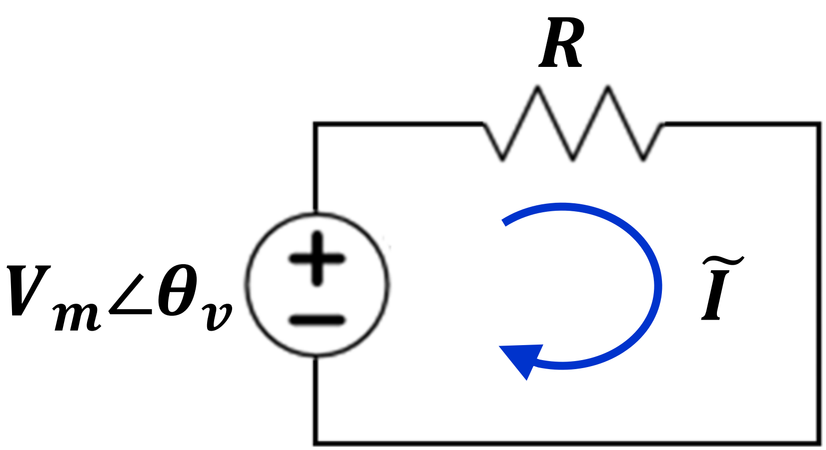
$$\frac{\tilde{V}}{\tilde{I}} = \frac{1}{\mathrm{j}\omega C} = -\mathrm{j}\,\frac{1}{\omega C}$$

Resistor



$$Z_R = \frac{\tilde{V}}{\tilde{I}} = R$$

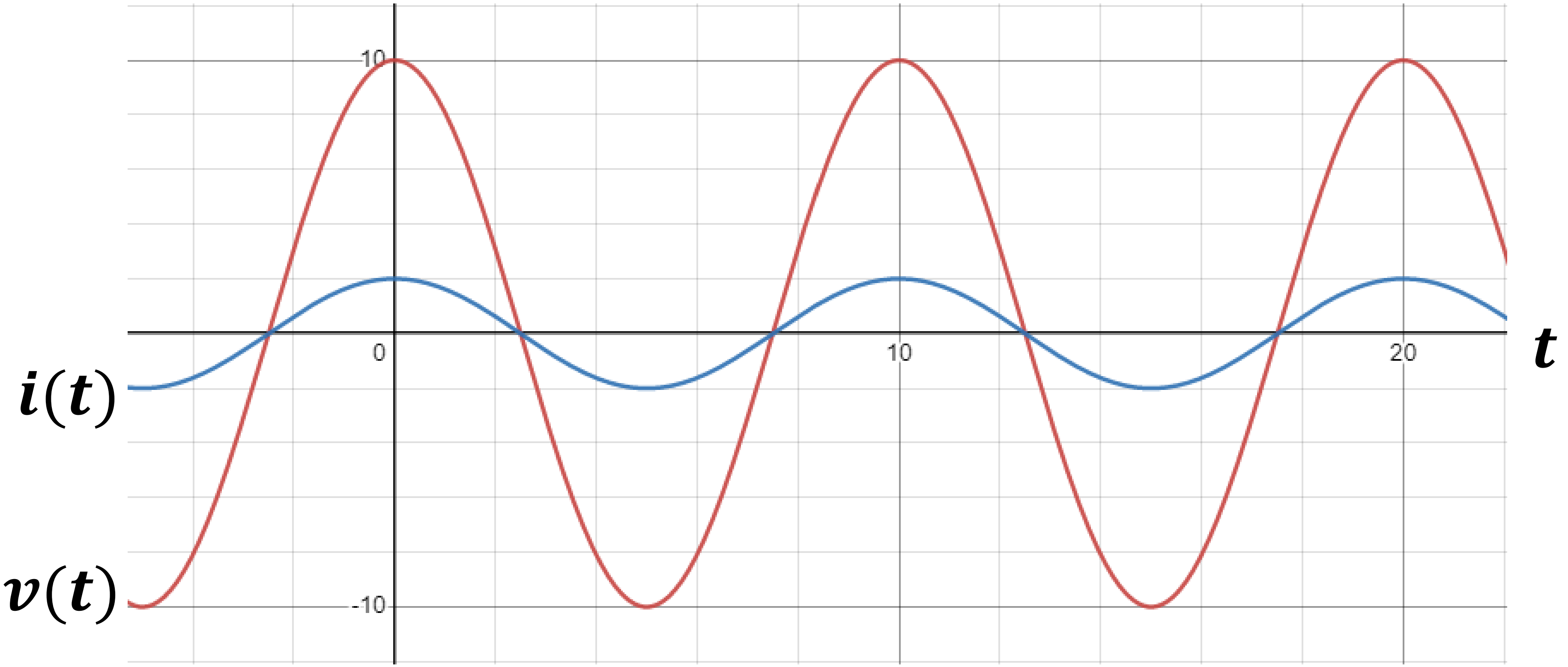
Impedance of resistor



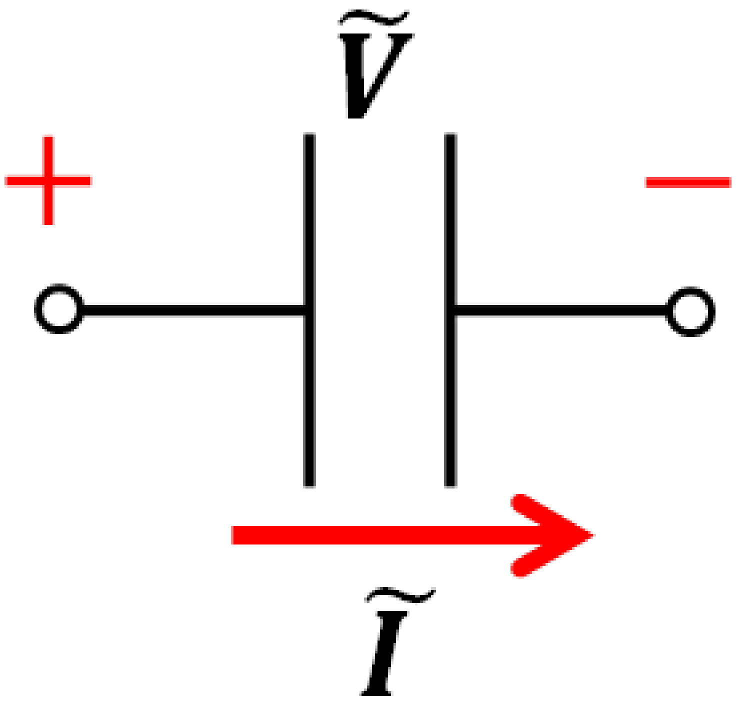
$$\tilde{I} = \frac{V_m \angle \theta_v}{R}$$

Current and voltage are in phase

Current and voltage are in phase in a resistor

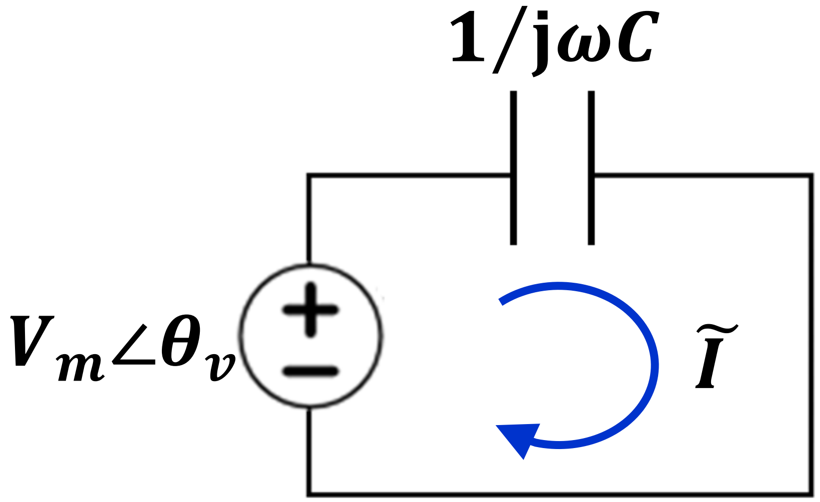


Impedance of Capacitor



$$Z_C = \frac{\tilde{V}}{\tilde{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{1}{\mathrm{j}\omega C} = -j\frac{1}{\omega C}$$

Capacitor



$$\tilde{I} = \frac{V_m\angle\theta_v}{\frac{1}{\mathrm{j}\omega C}} = \mathrm{j}\omega C\, V_m\angle\theta_v = \omega C\, V_m\angle(\theta_v + \pi/2)$$

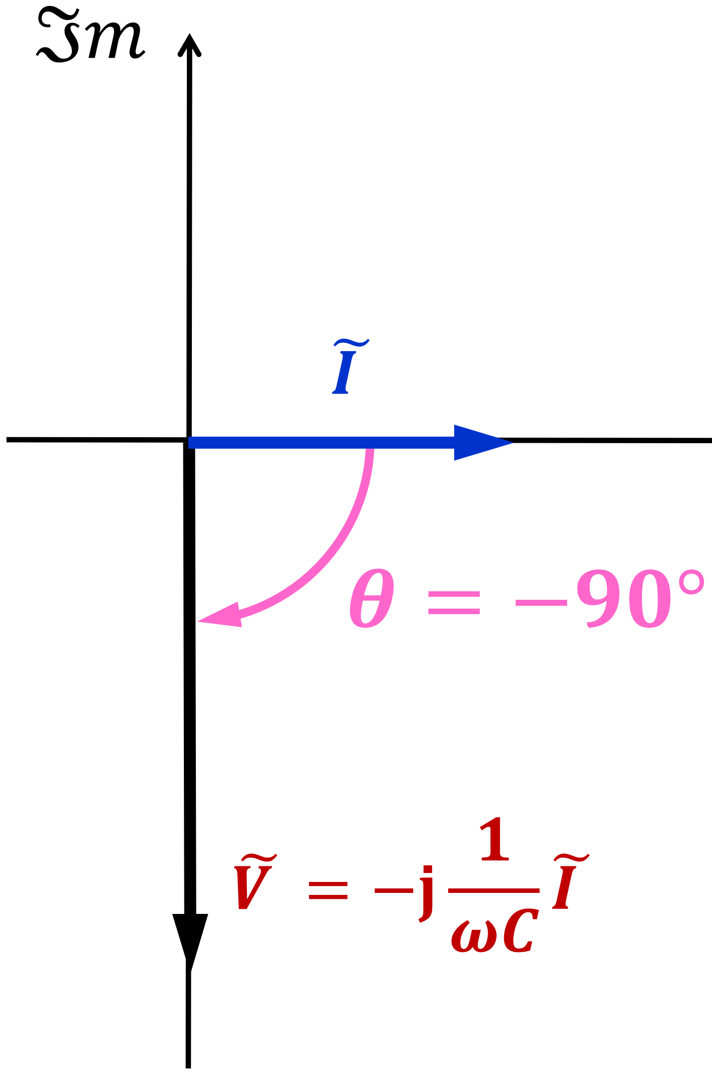
$$\theta_i = \theta_v + \frac{\pi}{2}$$

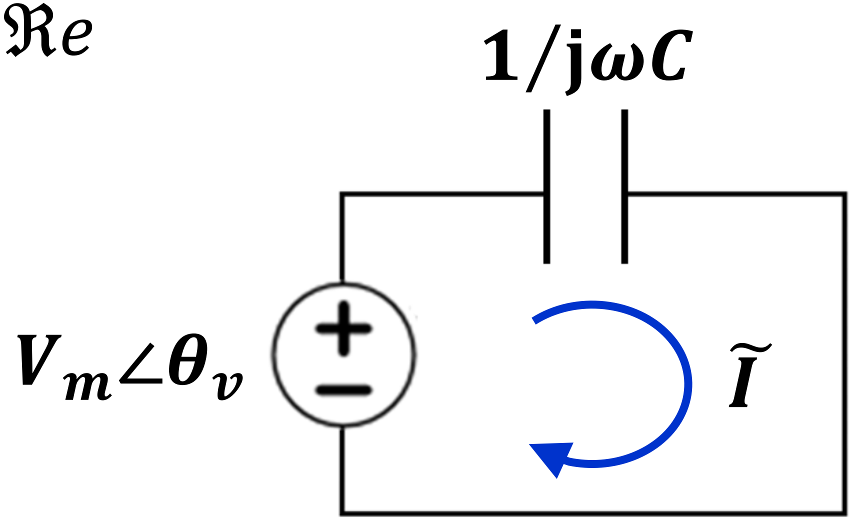
$$j = 1 \cdot \angle\frac{\pi}{2}$$

Current LEADS voltage by 90°

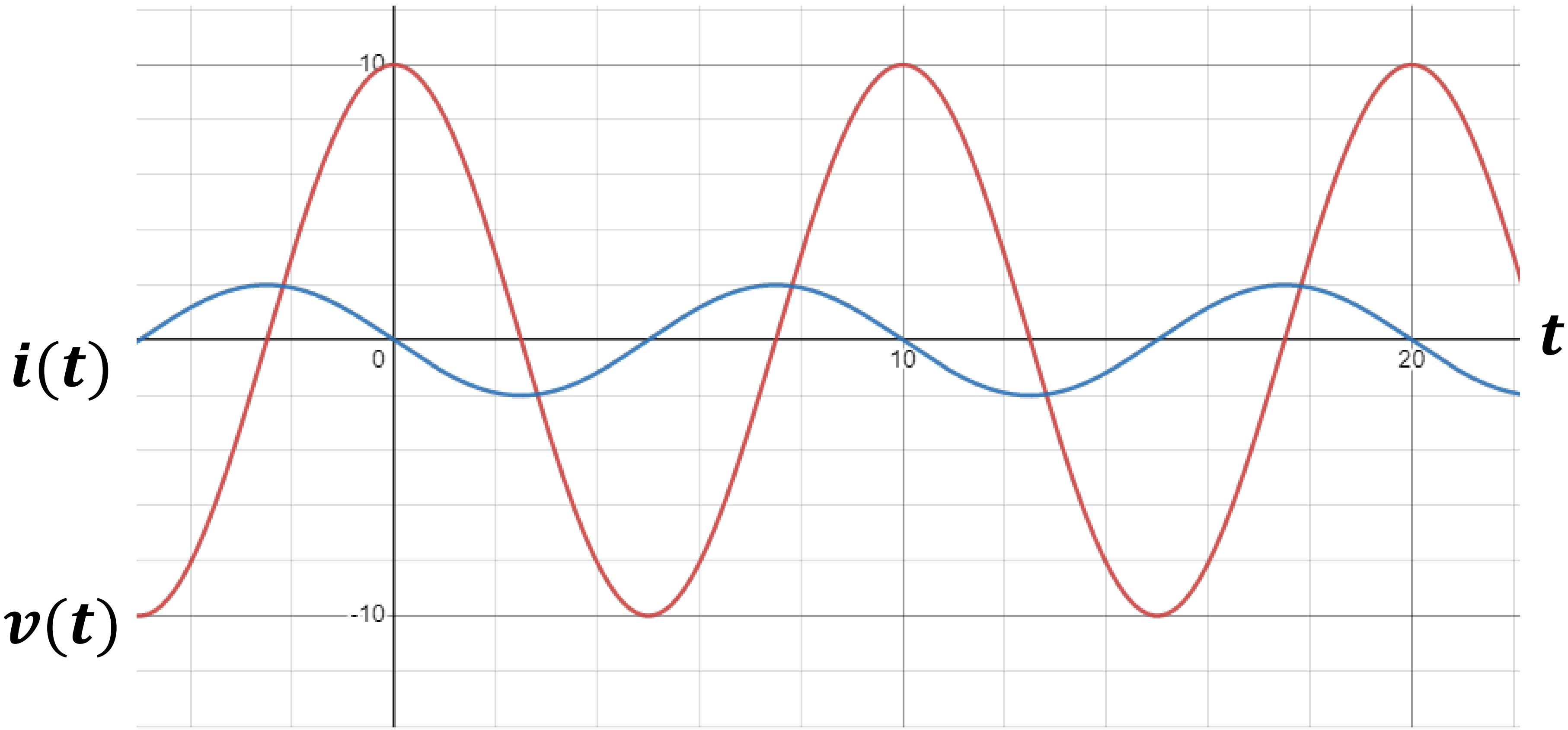
Pure capacitive reactance: current LEADS voltage by 90°

$$\tilde{V} = Z\,\tilde{I} = -\mathrm{j}\frac{1}{\omega C}\tilde{I}$$





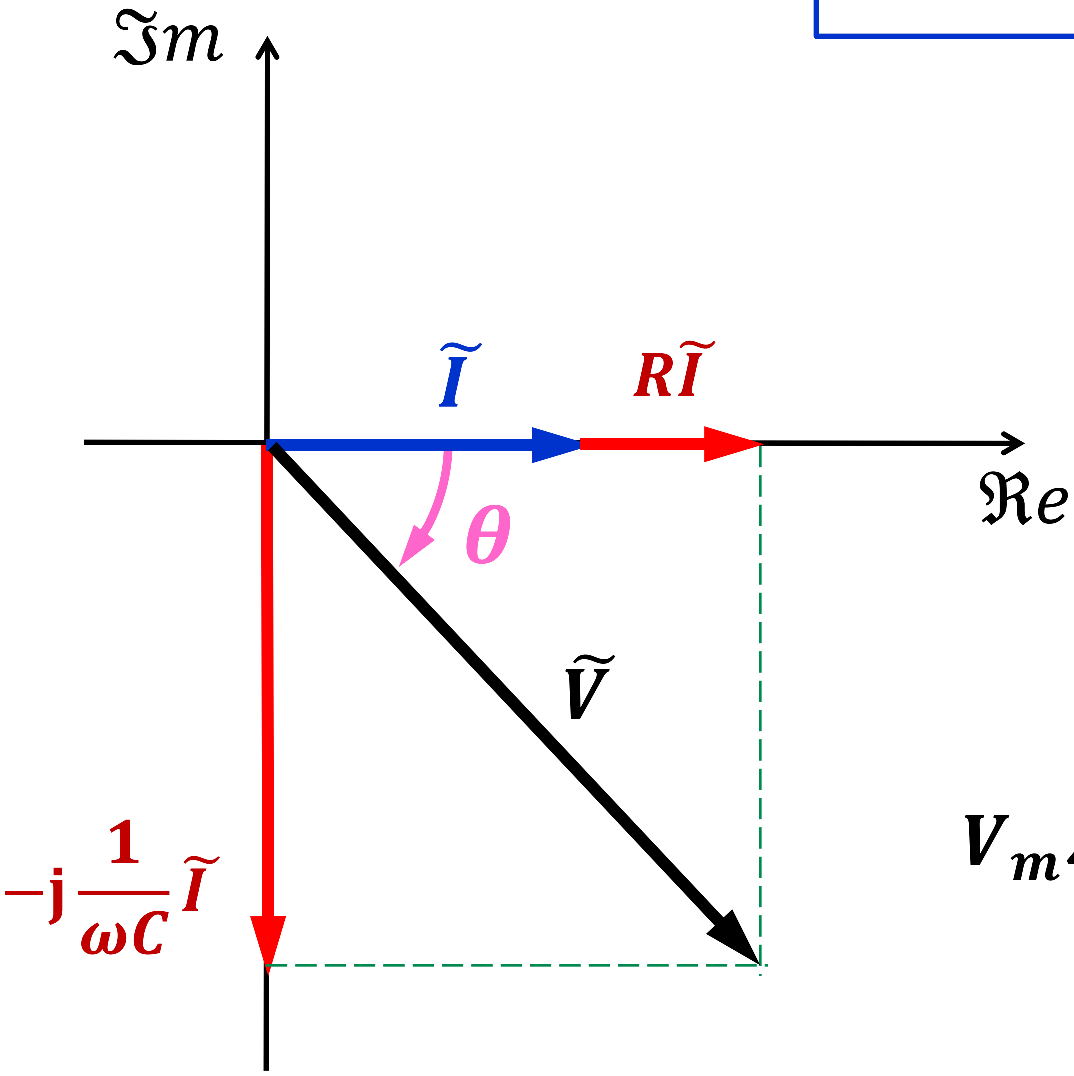
Current LEADS voltage by 90°
(it reaches peak value earlier)

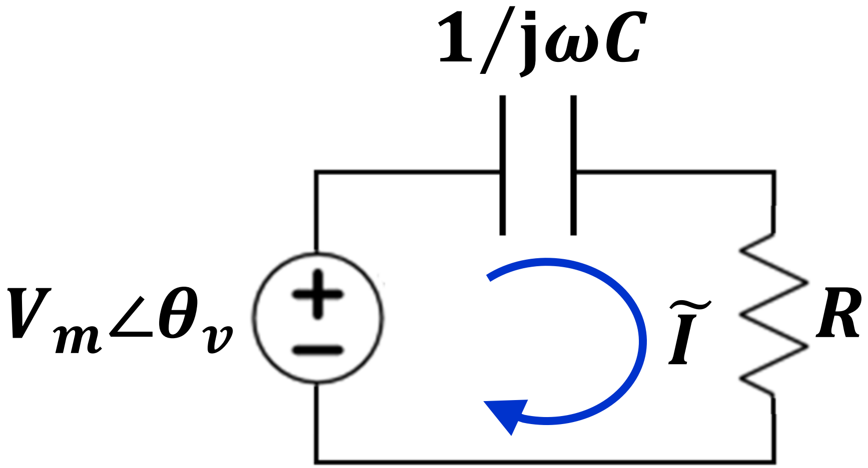


Waveforms are in "quadrature"

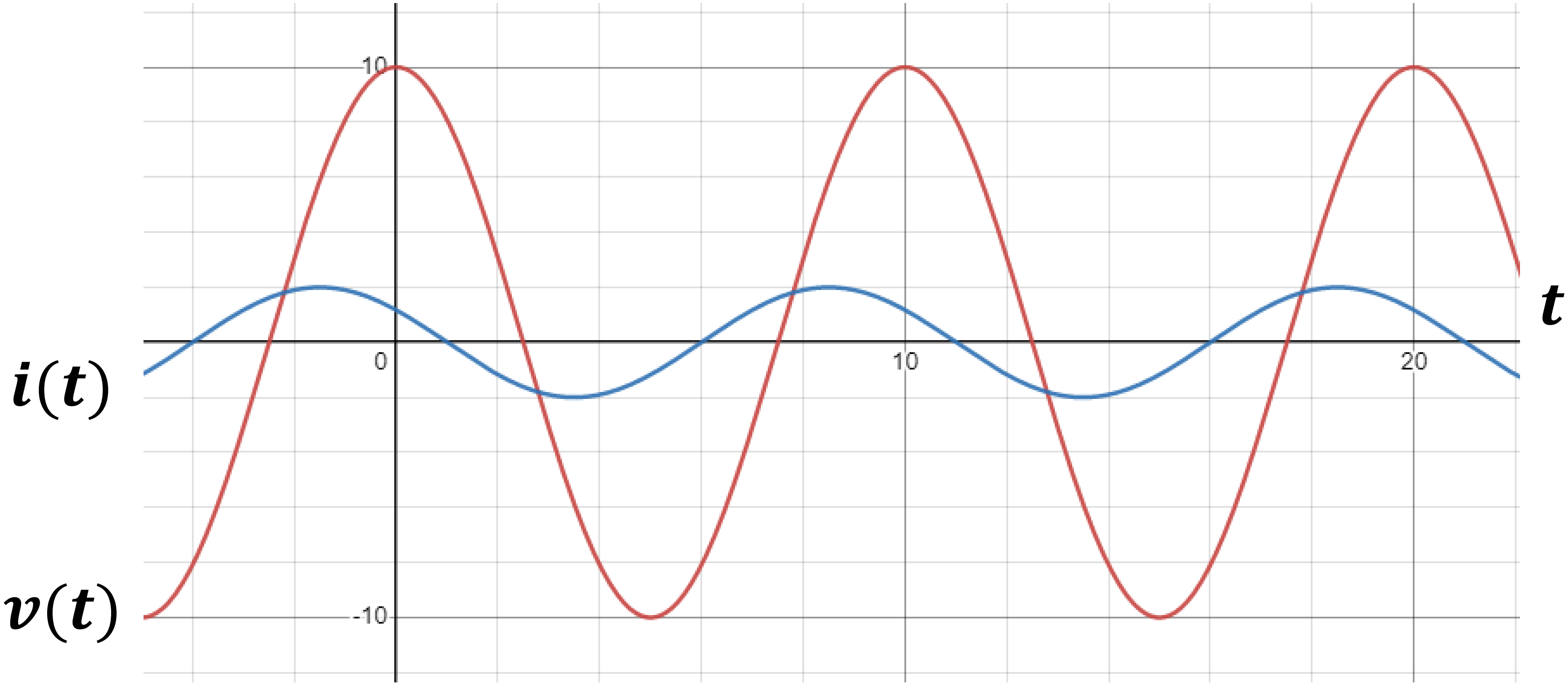
Total reactance is capacitive: current LEADS voltage

$$\tilde{V} = Z\,\tilde{I} = R\tilde{I} - \mathrm{j}\frac{1}{\omega C}\tilde{I}$$



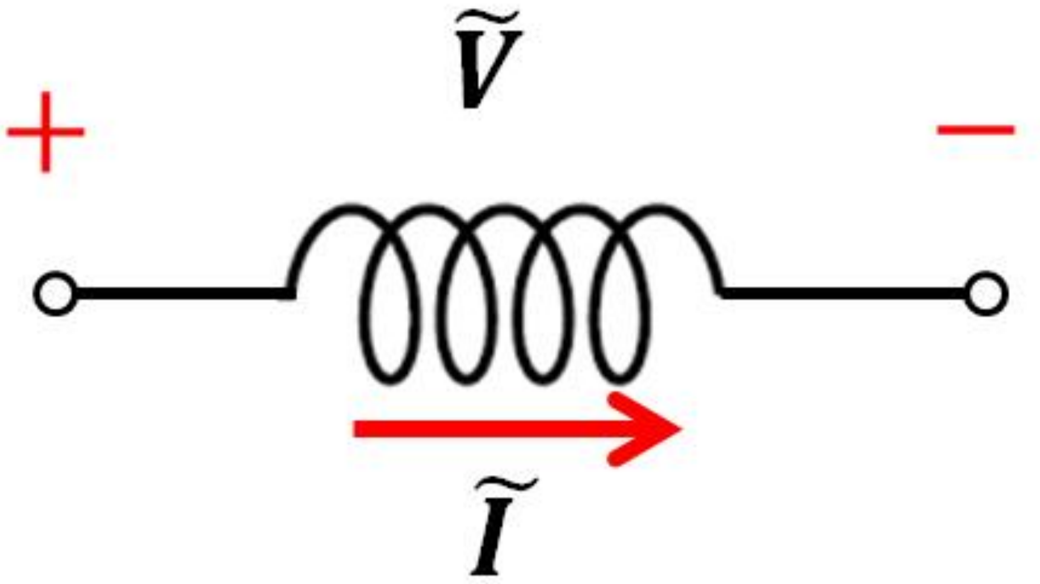


Current LEADS voltage
(it reaches peak value earlier)



But waveforms are NOT in "quadrature"

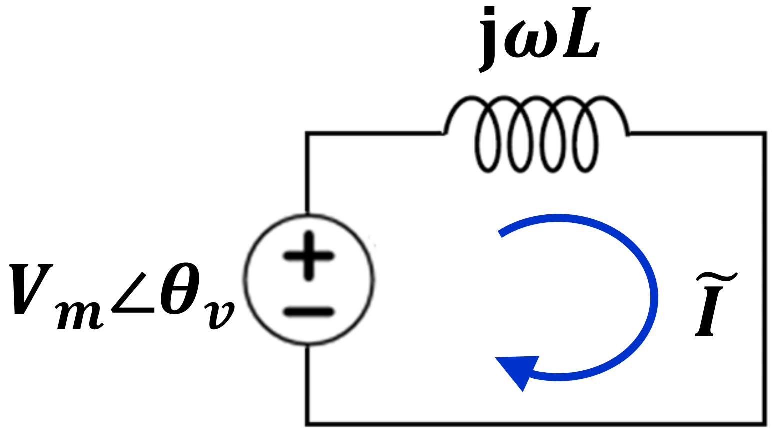
Impedance of Inductor



$$v_L(t) = L\frac{di_L(t)}{dt} \quad \Leftrightarrow \quad \tilde{V} = \mathrm{j}\omega L\tilde{I}$$

$$Z_L = \frac{\tilde{V}}{\tilde{I}} = \frac{V_m\angle\theta_v}{I_m\angle\theta_i} = \mathrm{j}\omega L$$

Inductor



$$\tilde{I} = \frac{V_m\angle\theta_v}{\mathrm{j}\omega L} = -\mathrm{j}\frac{V_m\angle\theta_v}{\omega L} = \frac{V_m}{\omega L}\angle(\theta_v - \pi/2)$$

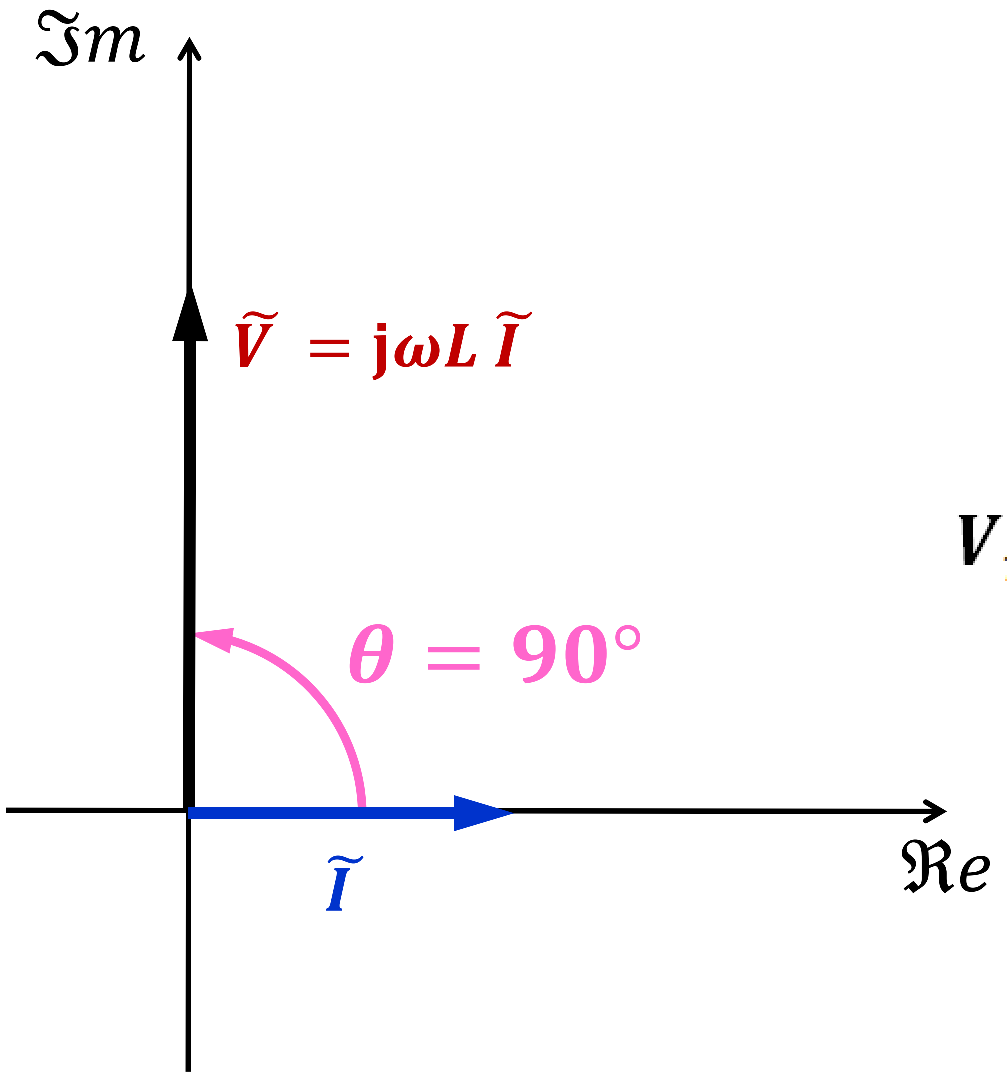
$$\theta_i = \theta_v - \frac{\pi}{2}$$

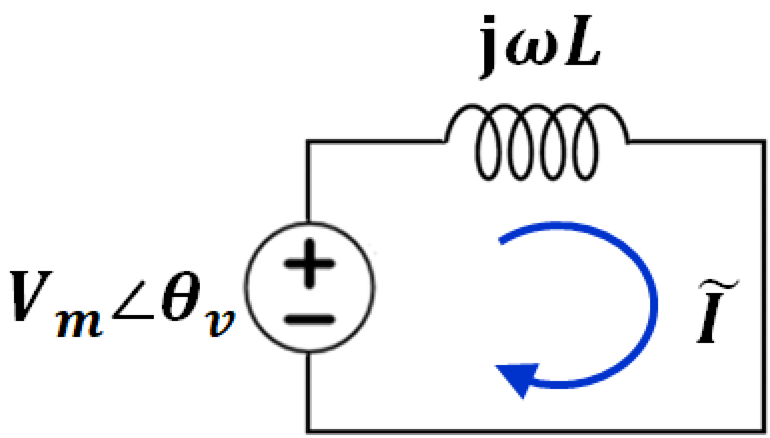
$$-\mathrm{j} = 1\cdot\angle-\frac{\pi}{2}$$

Current LAGS voltage by 90°

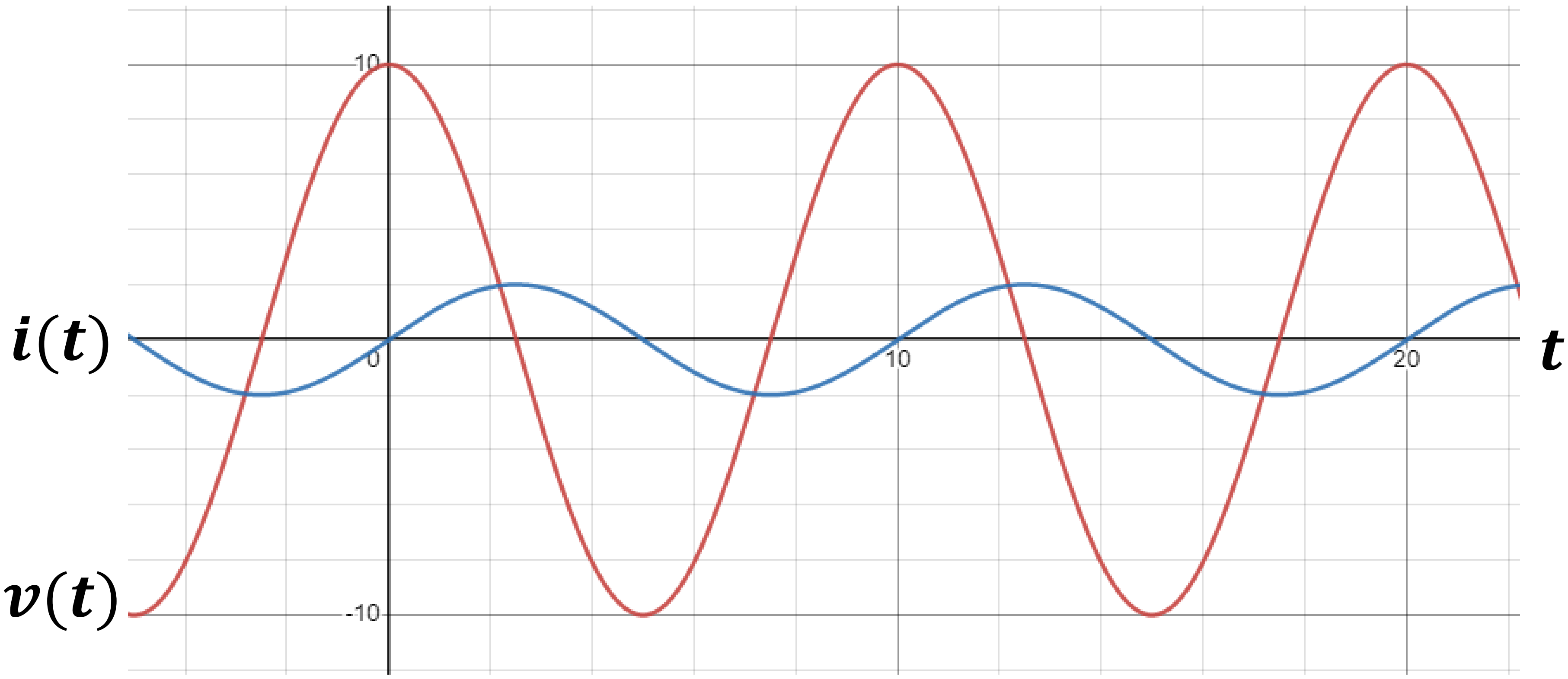
Pure inductive reactance: current LAGS voltage by 90°

$$\tilde{V} = Z\,\tilde{I} = j\omega L\,\tilde{I}$$





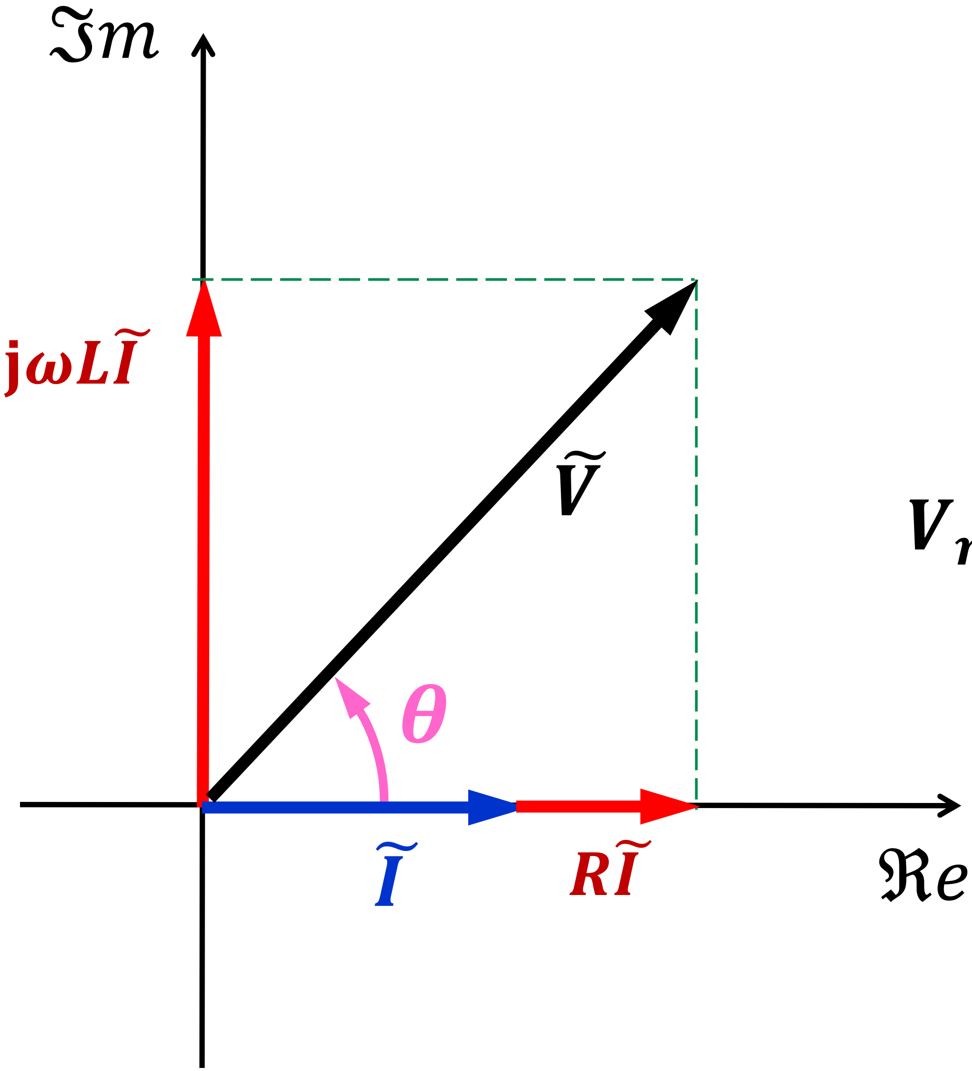
Current LAGS voltage by 90°
(it reaches peak value later)

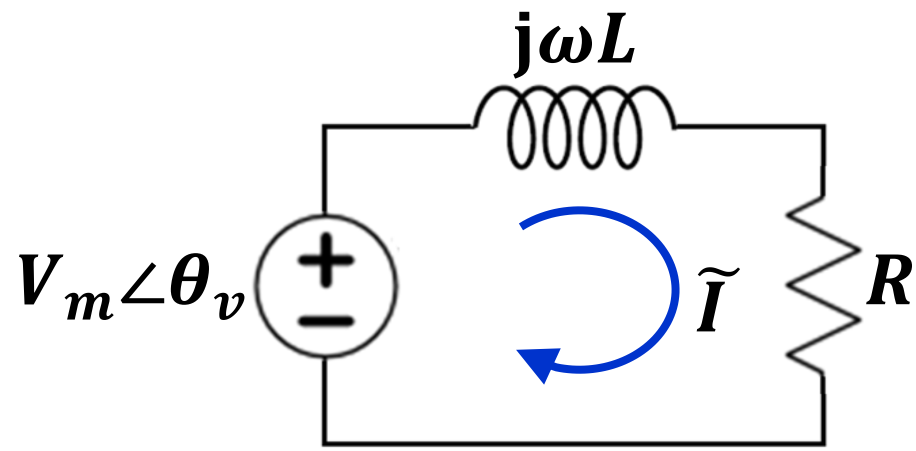


Waveforms are in "quadrature"

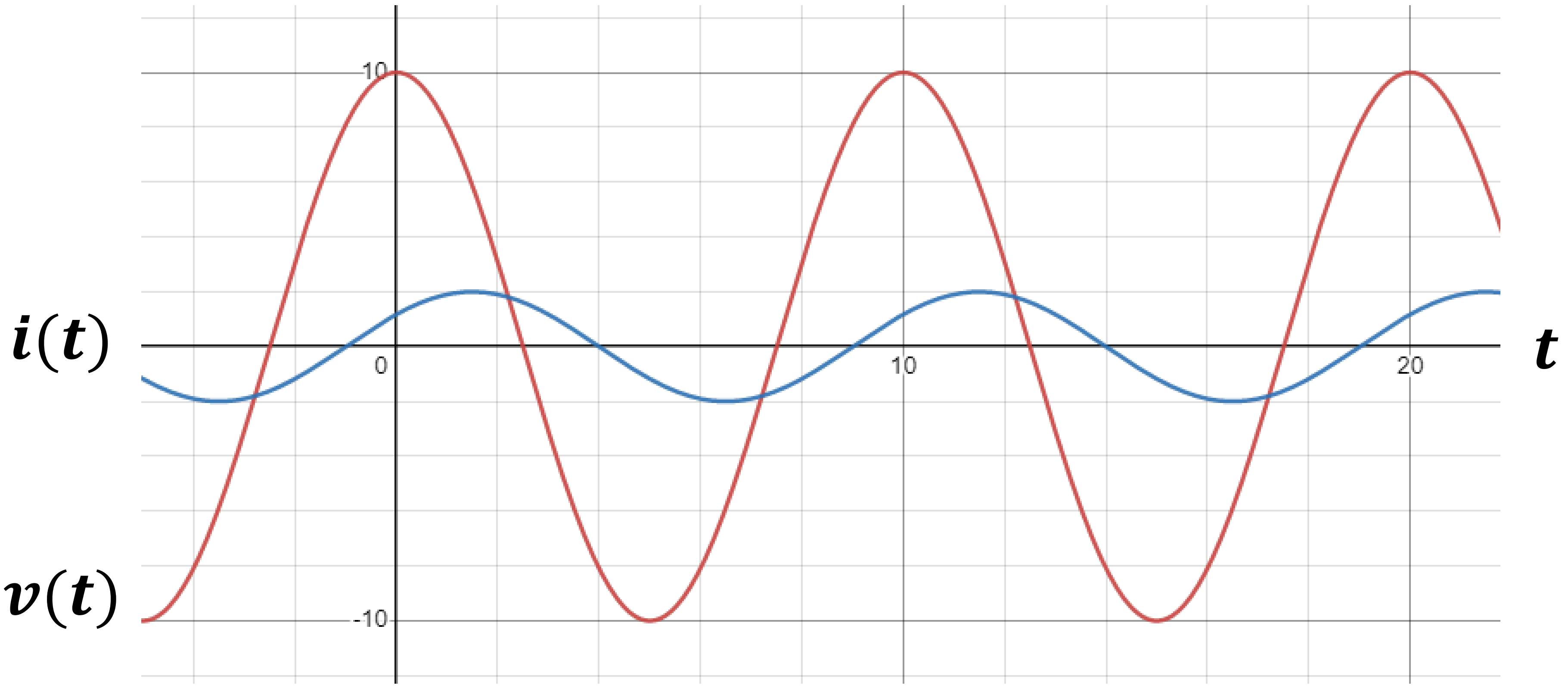
Total reactance is inductive: current LAGS voltage

$$\tilde{V} = Z\,\tilde{I} = R\tilde{I} + \mathrm{j}\omega L\,\tilde{I}$$





Current LAGS voltage
(it reaches peak value later)

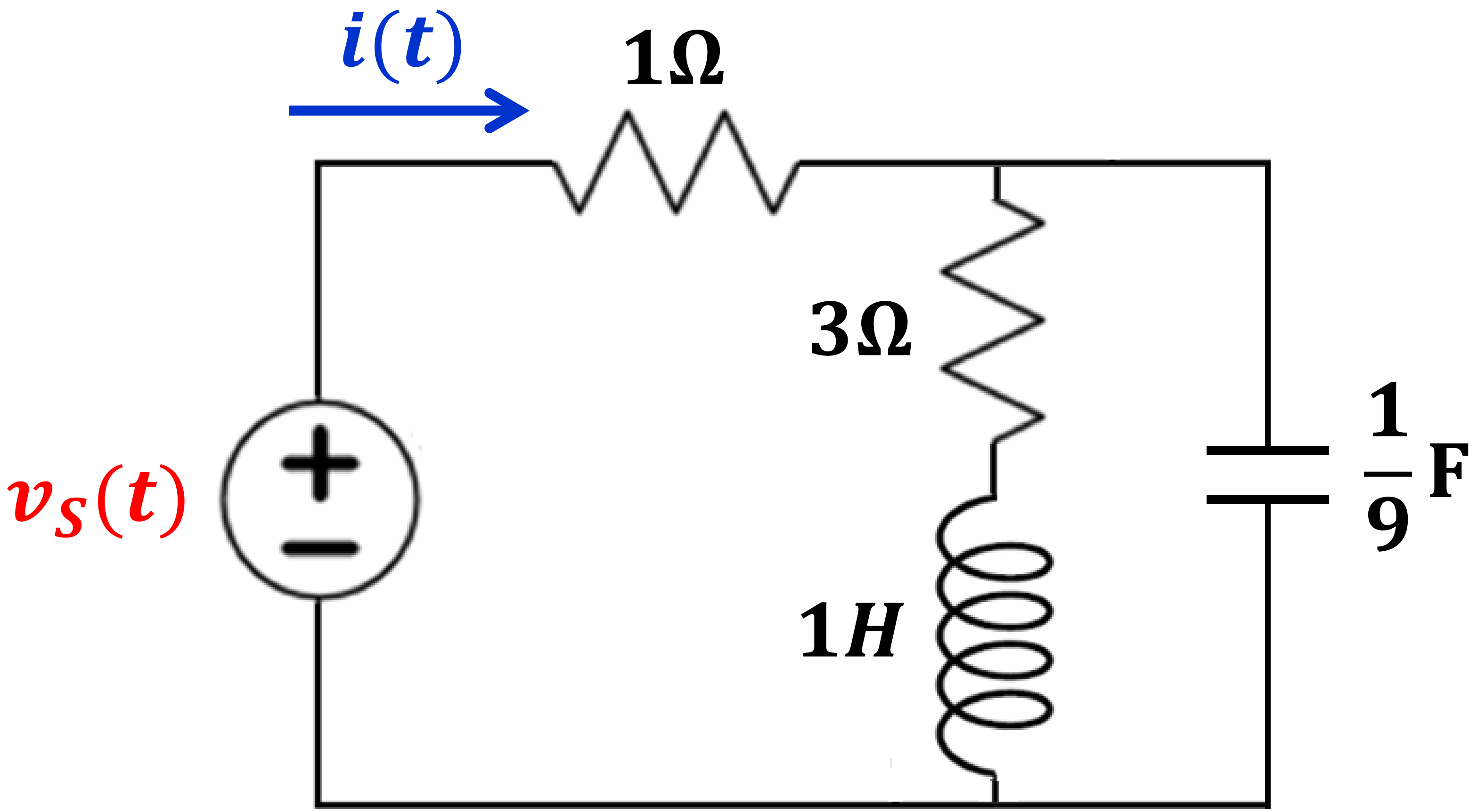


But waveforms are NOT in "quadrature"

Example of Phasor Analysis

$$v_S(t) = 10\cos(3t)$$
$$\omega = 3\ \text{rad/s}$$

Determine $i(t)$

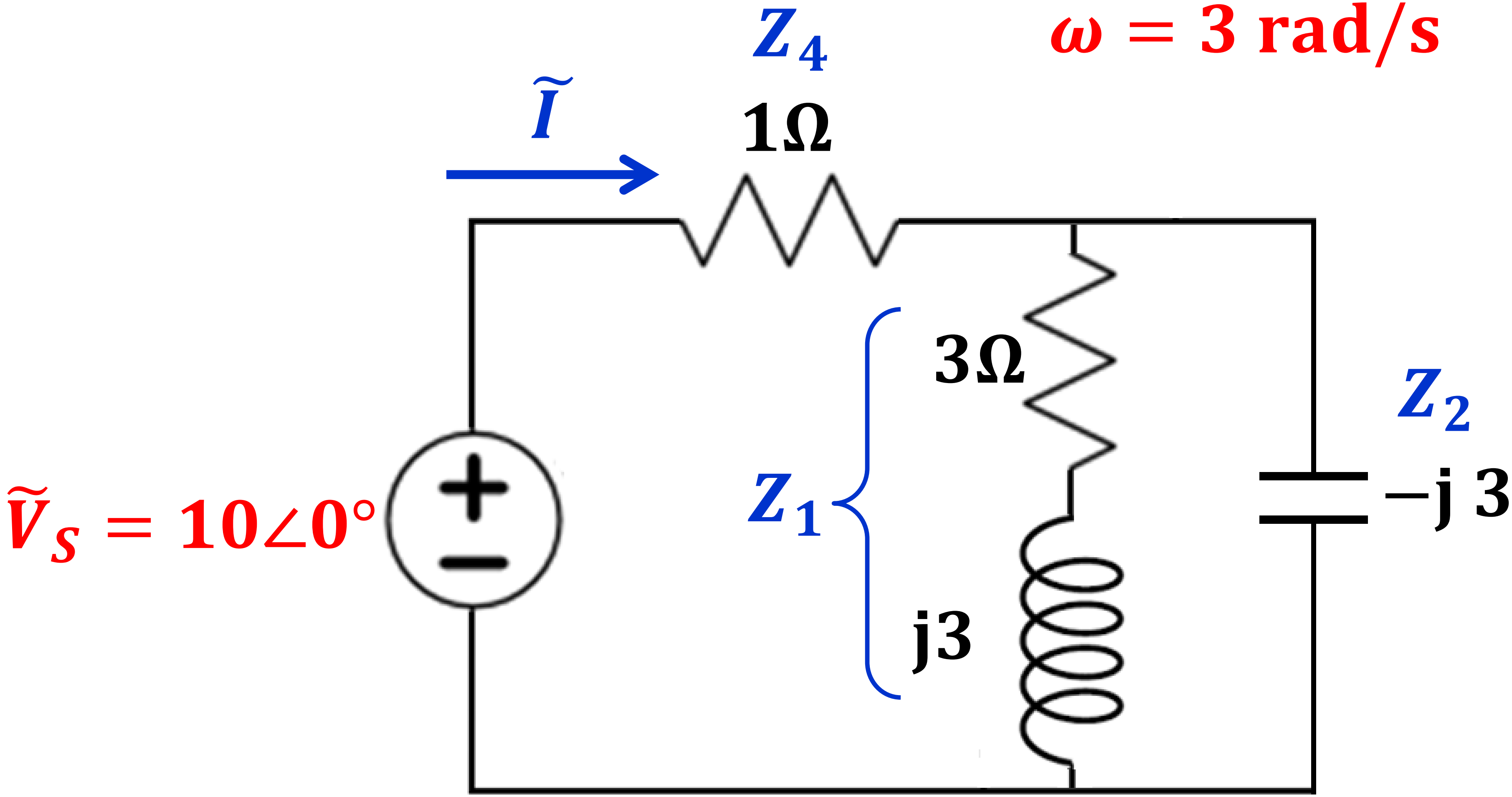


$$\text{j}\omega L = \text{j} \times 3 \times 1H = \text{j}3\ \Omega$$
$$1/\text{j}\omega C = -\text{j}/(3 \times 1/9\ \text{F}) = -\text{j}3\ \Omega$$

Phasor form

$v_S(t) = 10\cos(3t)$

$\omega = 3\ \text{rad/s}$



$Z_1 = 3 + j3\ \Omega$

$Z_2 = -j3\ \Omega$

$Z_3 = Z_1 // Z_2$

$$Z_1 = 3 + j3\ \Omega \qquad Z_2 = -j3\ \Omega$$

$$Z_3 = Z_1//Z_2 = \left(\frac{1}{3+j3} + \frac{1}{-j3}\right)^{-1} =$$

$$= \left(\frac{\cancel{-j3} + 3 + \cancel{j3}}{-j3(3+j3)}\right)^{-1} = \left(\frac{3}{9-j9}\right)^{-1} = (3-j3)\Omega$$

$$Z_{eq} = Z_4 + Z_1//Z_2 = 1 + 3 - j3 = (4-j3)\Omega$$

$$\tilde{I} = \frac{\tilde{V}_S}{Z_{eq}} = \frac{10\angle 0^\circ}{(4-j3)}\ \mathrm{A}$$

$$\tilde{I} = \frac{\tilde{V}_S}{Z_{eq}} = \frac{10\angle 0°}{(4 - j3)}\ \text{A}$$

$$\tilde{I} = \frac{10(4 + j3)}{(4 - j3)(4 + j3)} = \frac{40 + j30}{16 + 9 - \cancel{j12} + \cancel{j12}}$$

$$\tilde{I} = \frac{40 + j30}{25} = 1.6 + j1.2\ \text{A}$$

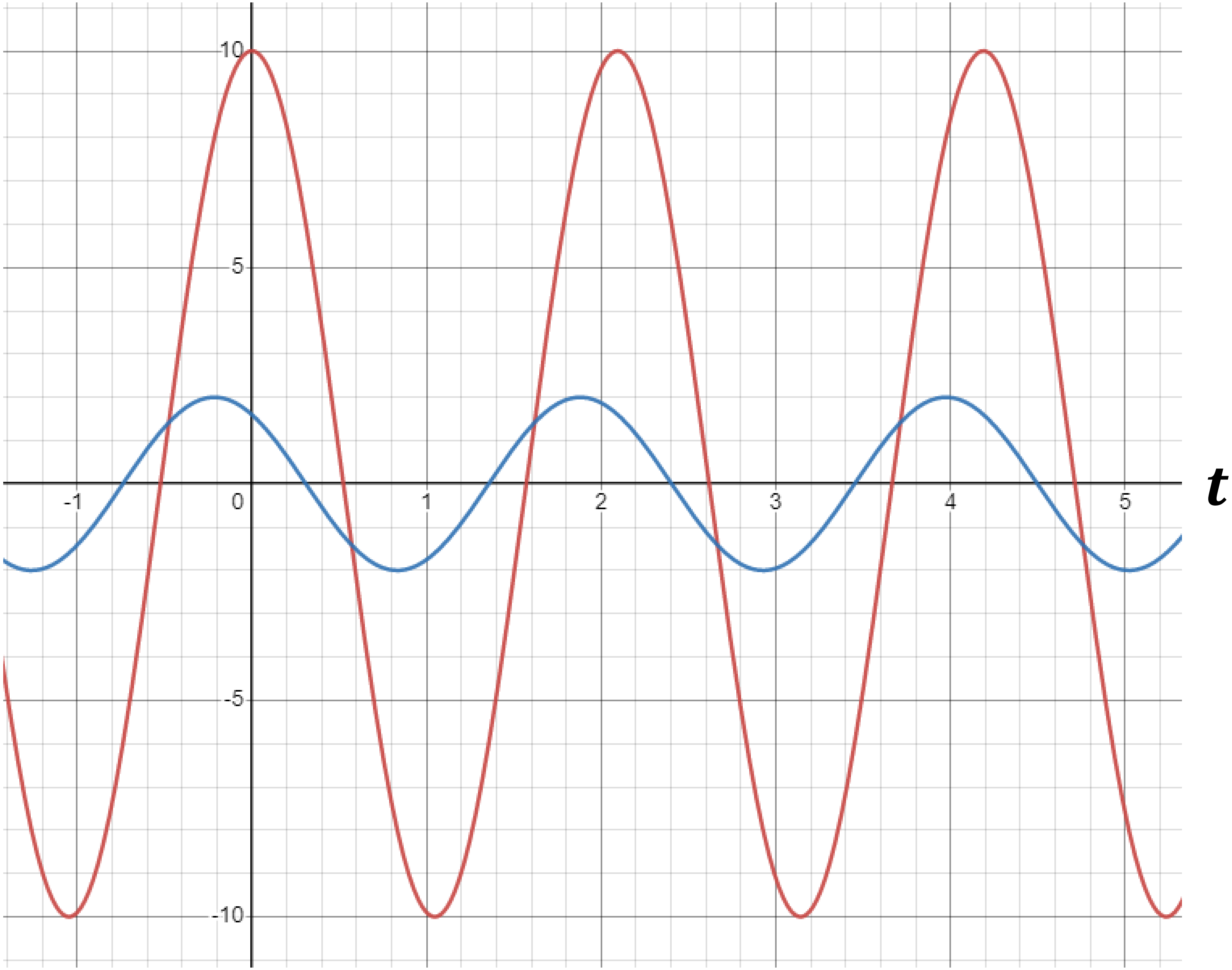
$$|\tilde{I}| = \sqrt{1.6^2 + 1.2^2} = \sqrt{4} = 2\ \text{A}$$

$$\angle \tilde{I} = \theta_I = \tan^{-1}\left(\frac{1.2}{1.6}\right) = 0.6435\ \text{rad} = 36.87°$$

$$\tilde{I} = 2\angle 36.87°\ \text{A} \Leftrightarrow$$

$$\Leftrightarrow\ i(t) = 2\cos(3t + 36.87°)\ \text{A}$$

$$v_S = 10\cos(3t)$$



$$i(t) = 2\cos(3t + 36.87°)\ \text{A}$$