ECE 205 “Electrical and Electronics Circuits”

Spring 2024 – LECTURE 20

MWF – 12:00pm

Prof. Umberto Ravaioli

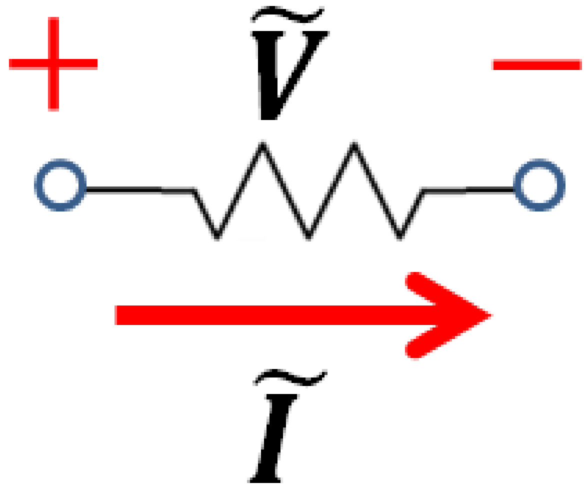
2062 ECE Building

Lecture 20 – Summary

Learning Objectives

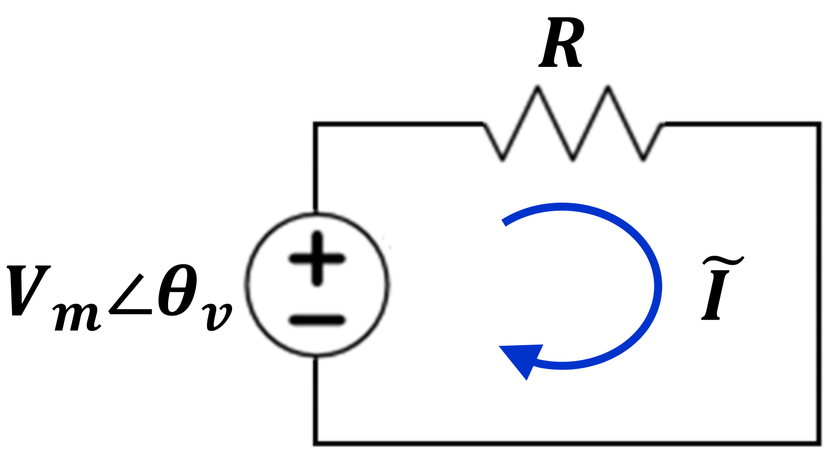
1. Solution of circuit problems with phasors

Resistor



$$Z_R = \frac{\widetilde{V}}{\widetilde{I}} = R$$

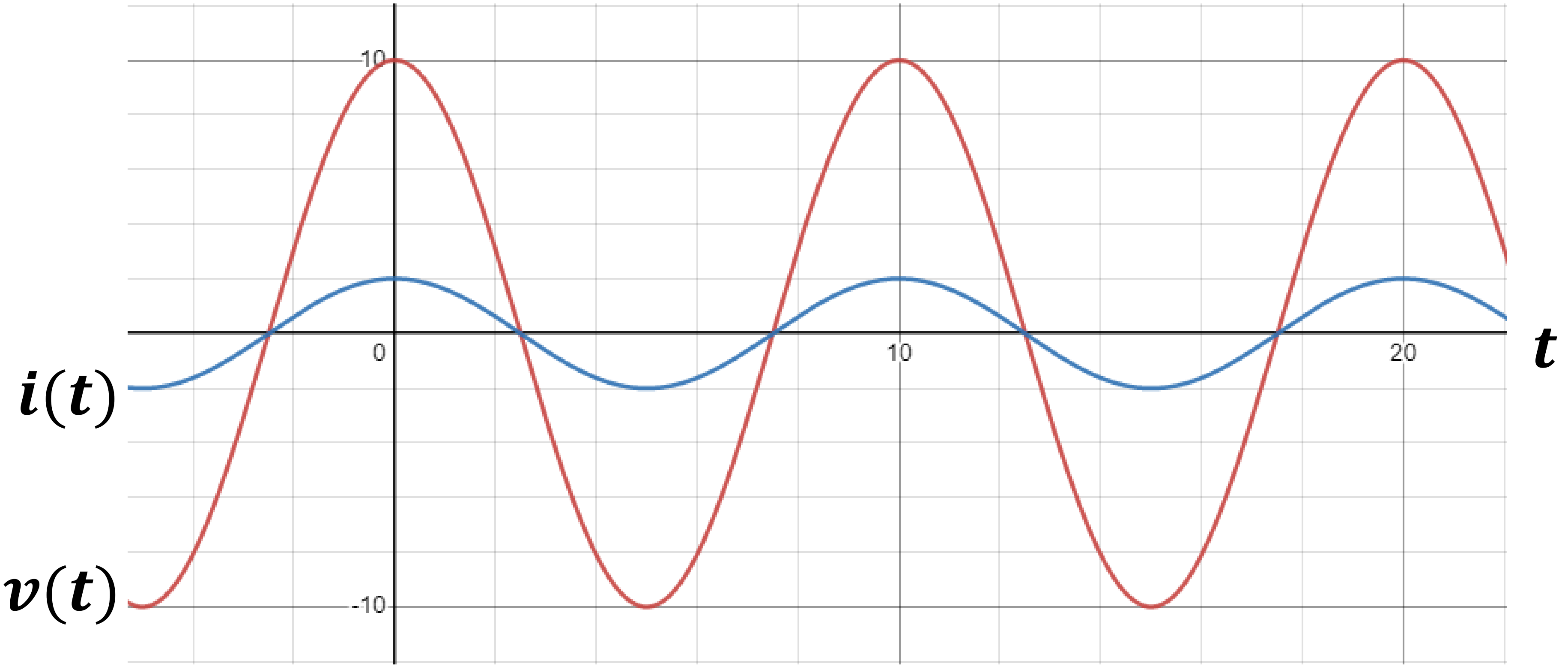
Impedance of resistor



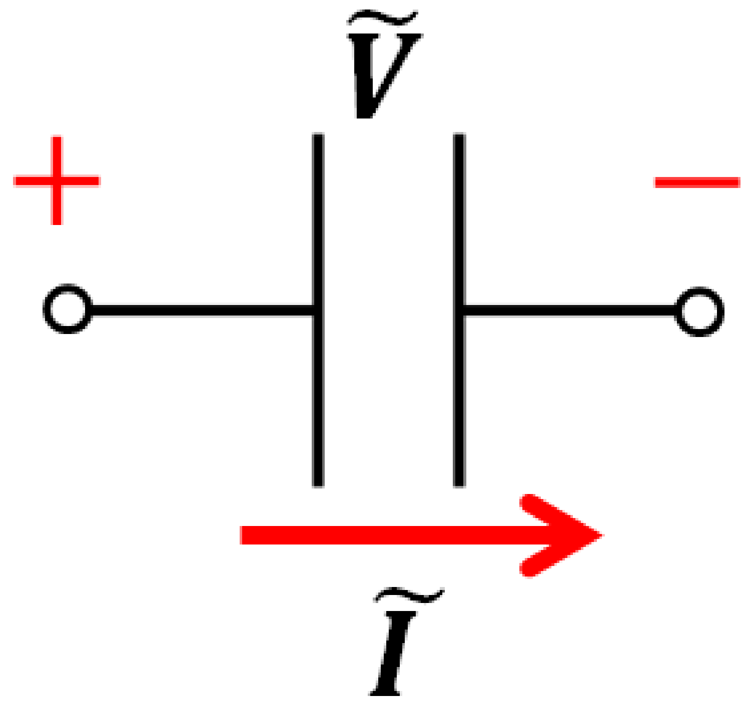
$$\widetilde{I} = \frac{V_m \angle \theta_v}{R}$$

Current and voltage are in phase

Current and voltage are in phase in a resistor

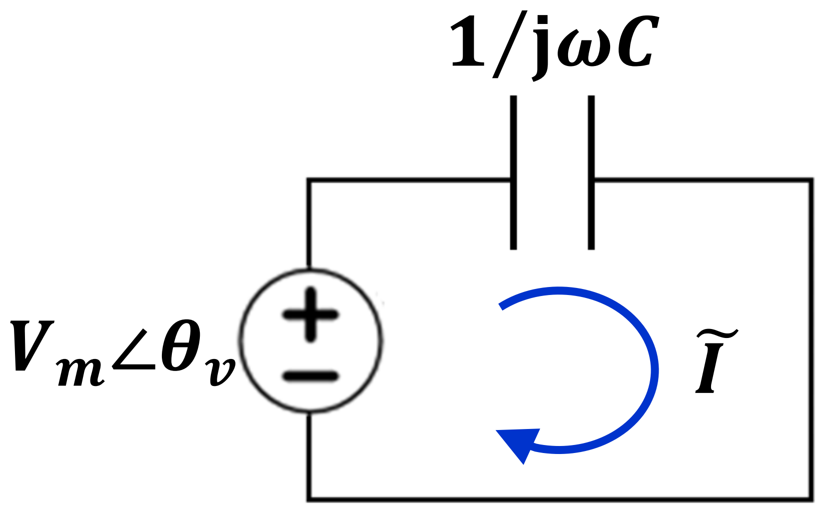


Impedance of Capacitor



$$Z_C = \frac{\tilde{V}}{\tilde{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{1}{\mathrm{j}\omega C} = -j\frac{1}{\omega C}$$

Capacitor



$$\tilde{I} = \frac{V_m\angle\theta_v}{\frac{1}{\mathrm{j}\omega C}} = \mathrm{j}\omega C\, V_m\angle\theta_v = \omega C\, V_m\angle(\theta_v + \pi/2)$$

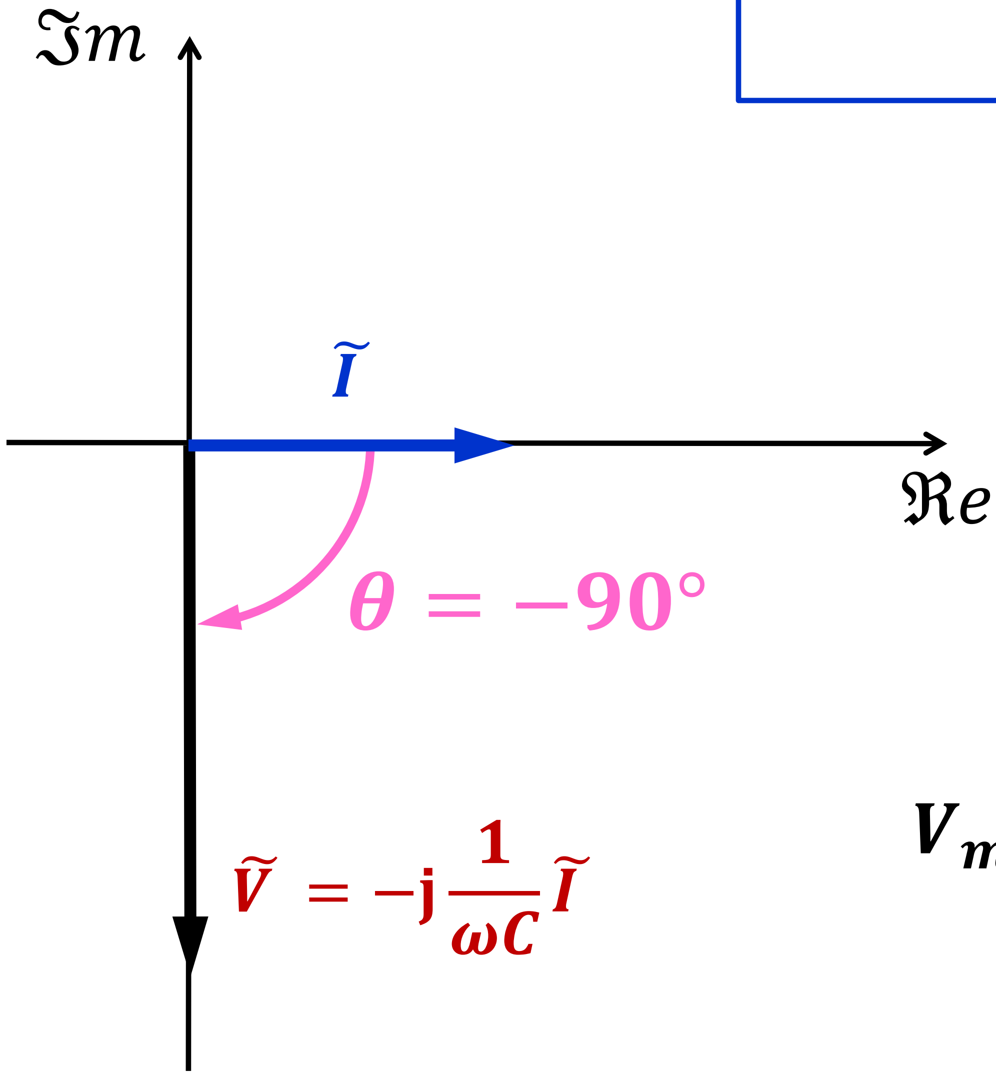
$$\theta_i = \theta_v + \frac{\pi}{2}$$

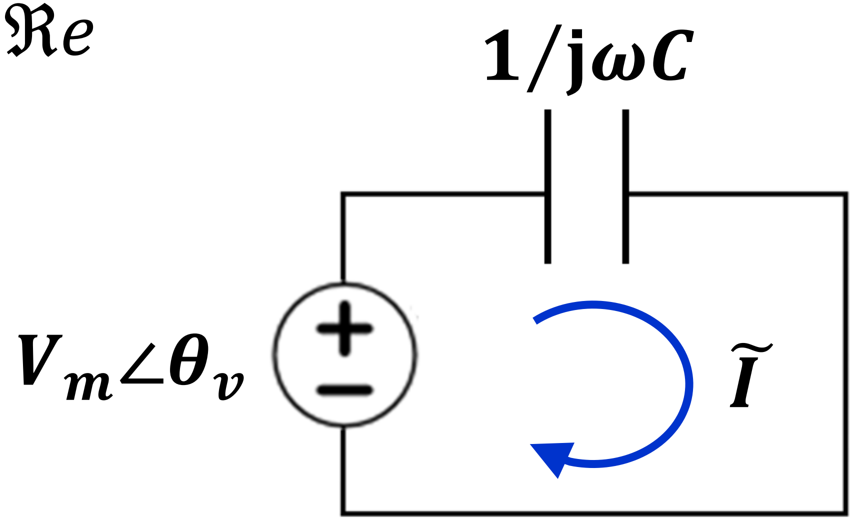
$$j = 1 \cdot \angle\frac{\pi}{2}$$

Current LEADS voltage by 90°

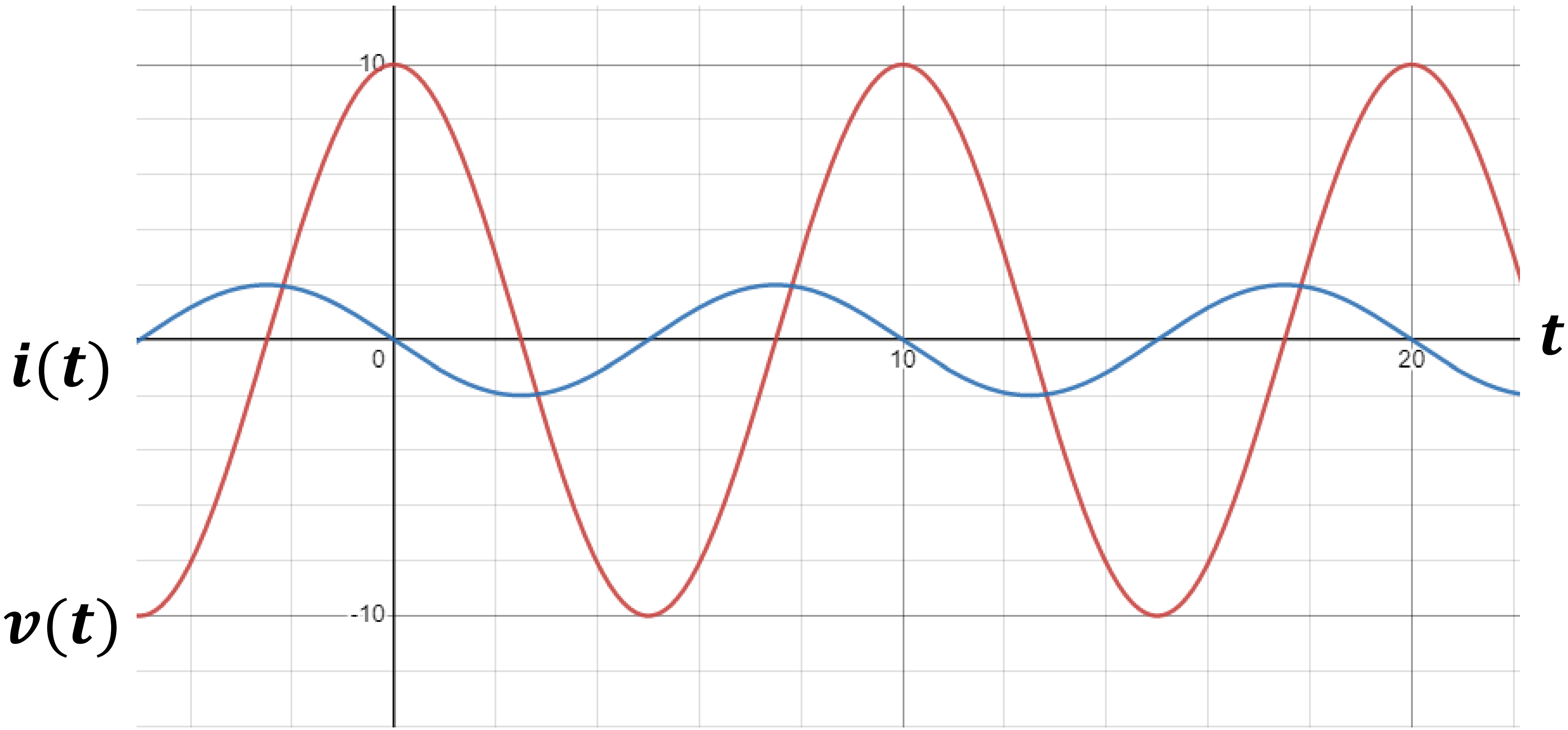
Pure capacitive reactance: current LEADS voltage by 90°

$$\tilde{V} = Z\,\tilde{I} = -\mathrm{j}\frac{1}{\omega C}\tilde{I}$$





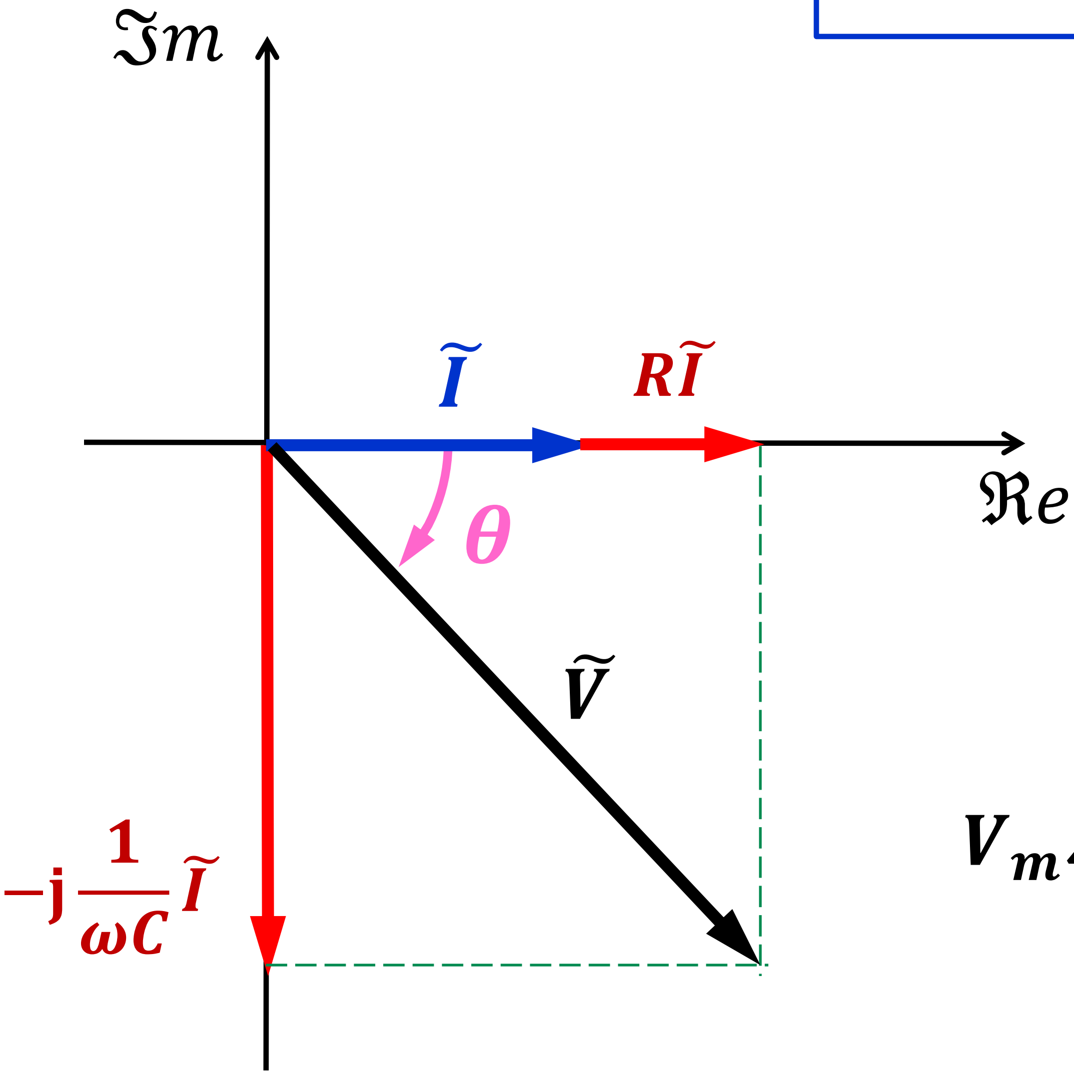
Current LEADS voltage by 90°
(it reaches peak value earlier)

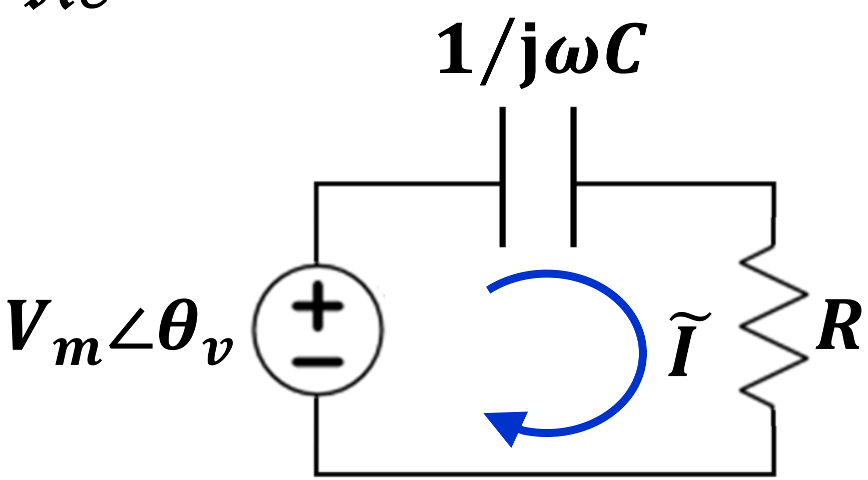


Waveforms are in "quadrature"

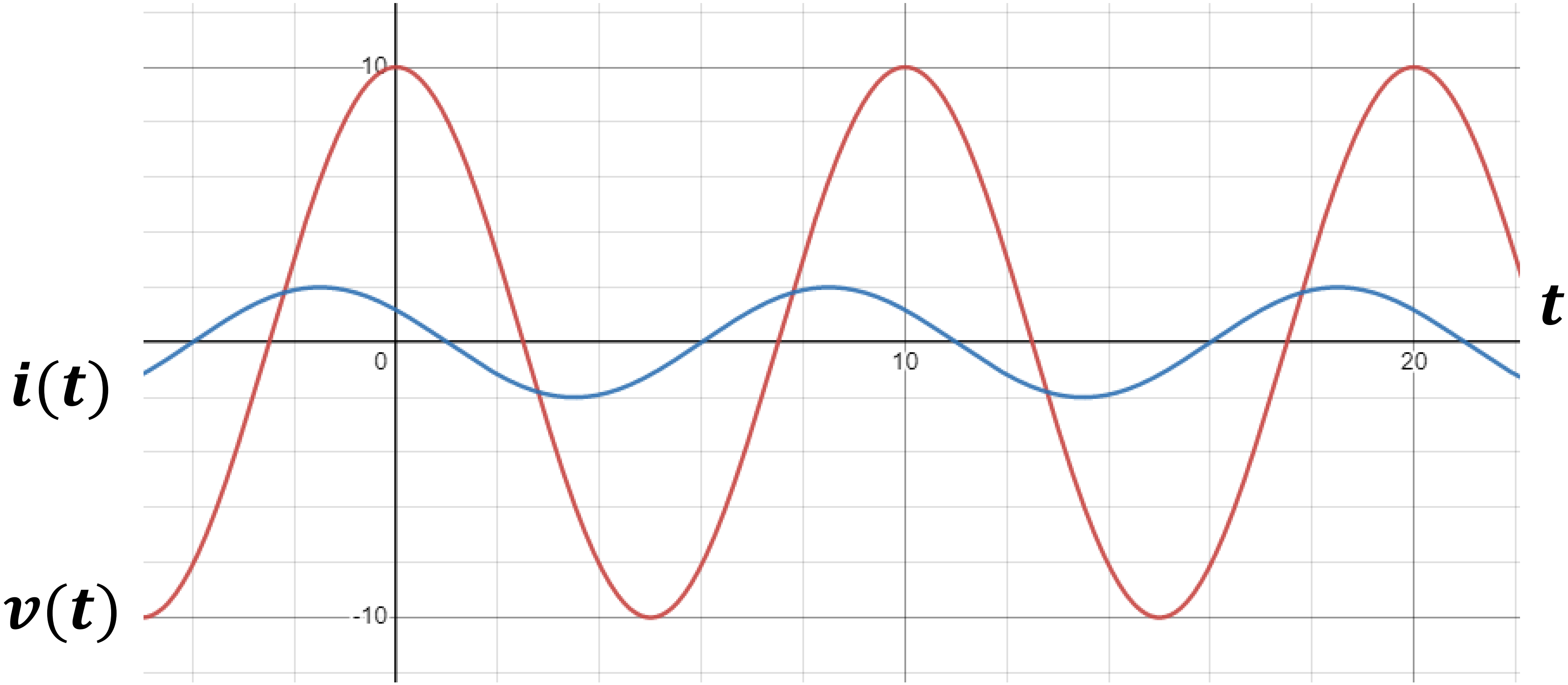
Total reactance is capacitive: current LEADS voltage

$$\tilde{V} = Z\tilde{I} = R\tilde{I} - \mathrm{j}\frac{1}{\omega C}\tilde{I}$$



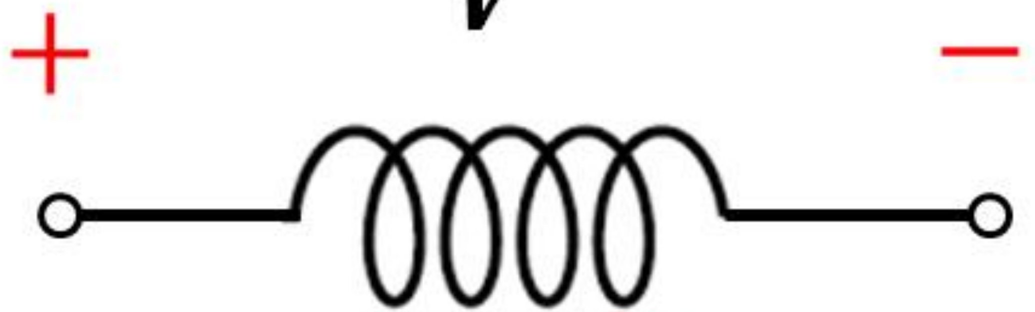


Current LEADS voltage
(it reaches peak value earlier)



But waveforms are NOT in "quadrature"

Impedance of Inductor



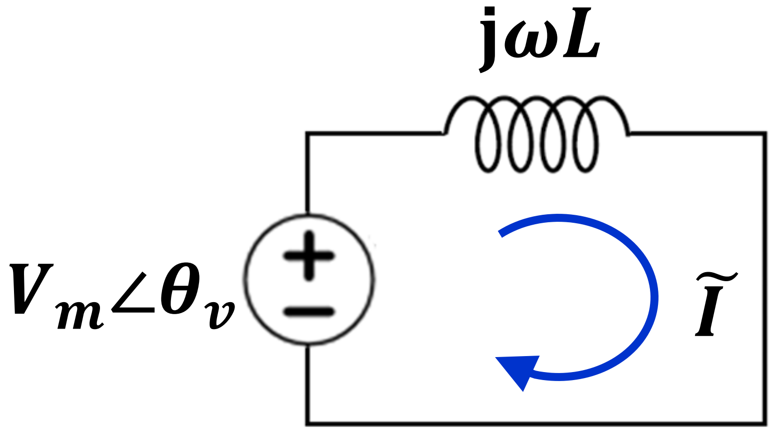
$$\tilde{V}$$

$$\tilde{I}$$

$$v_L(t) = L\frac{di_L(t)}{dt} \quad \Leftrightarrow \quad \tilde{V} = \mathrm{j}\omega L\tilde{I}$$

$$Z_L = \frac{\tilde{V}}{\tilde{I}} = \frac{V_m\angle\theta_v}{I_m\angle\theta_i} = \mathrm{j}\omega L$$

Inductor



$$\tilde{I} = \frac{V_m\angle\theta_v}{\mathrm{j}\omega L} = -\mathrm{j}\frac{V_m\angle\theta_v}{\omega L} = \frac{V_m}{\omega L}\angle(\theta_v - \pi/2)$$

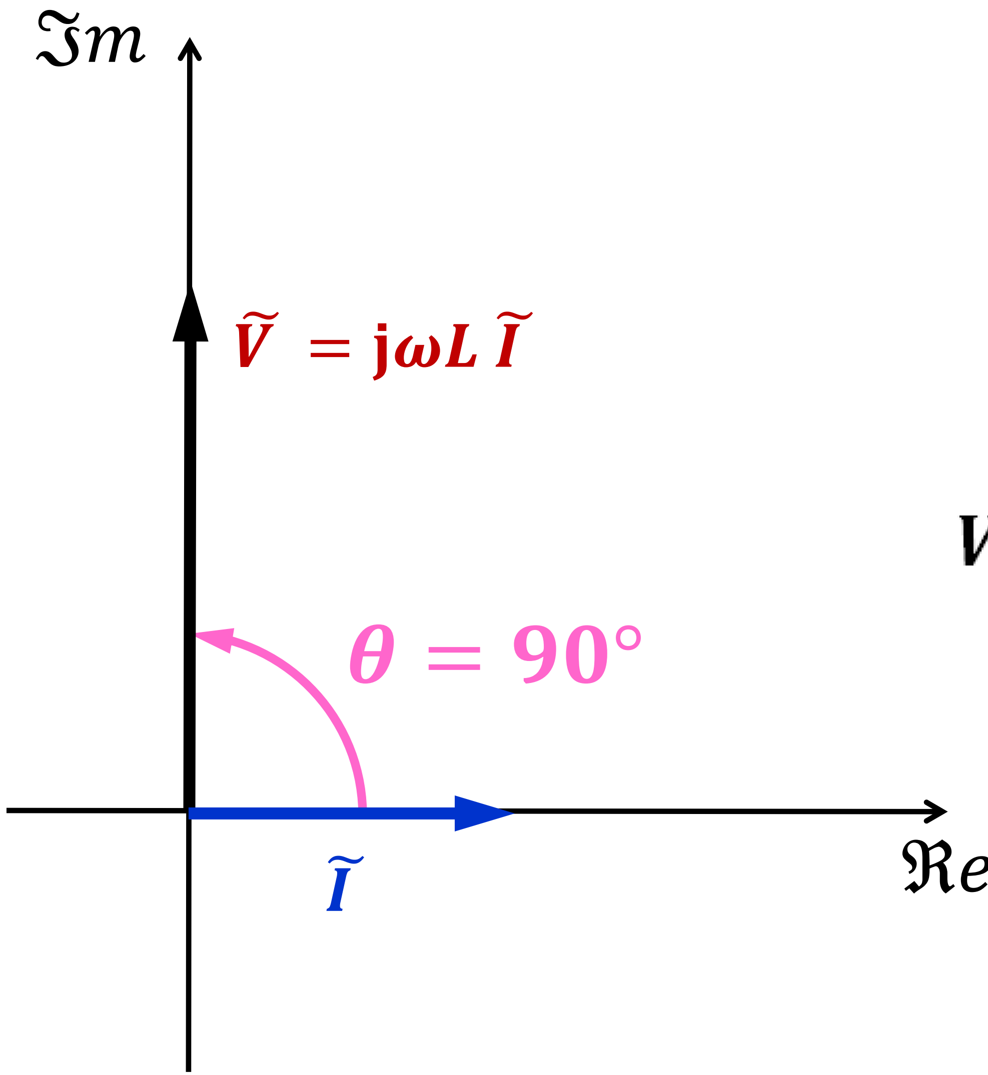
$$\theta_i = \theta_v - \frac{\pi}{2}$$

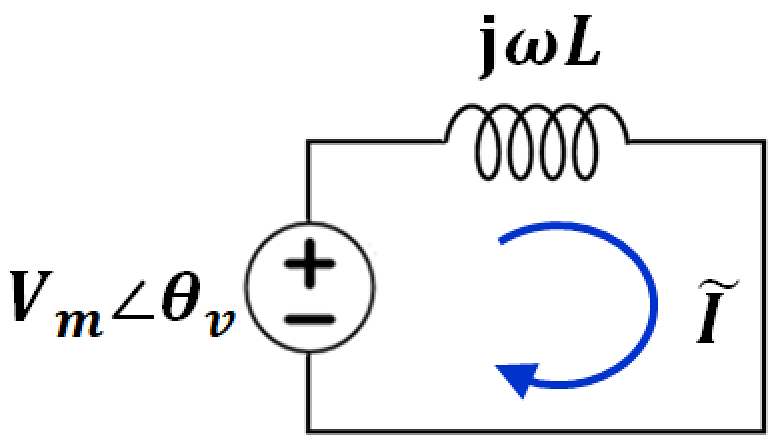
$$-\mathrm{j} = 1 \cdot \angle -\frac{\pi}{2}$$

Current LAGS voltage by 90°

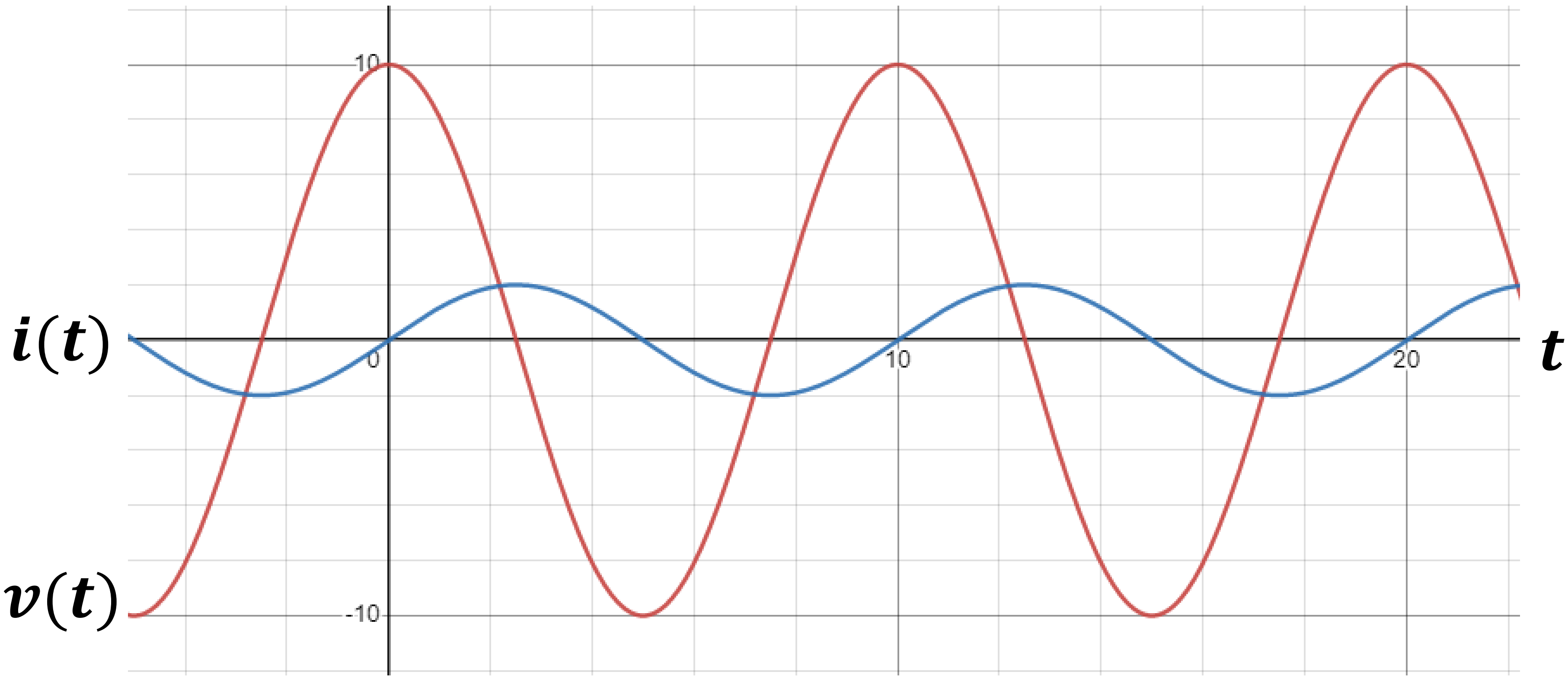
Pure inductive reactance: current LAGS voltage by 90°

$$\tilde{V} = Z\,\tilde{I} = j\omega L\,\tilde{I}$$





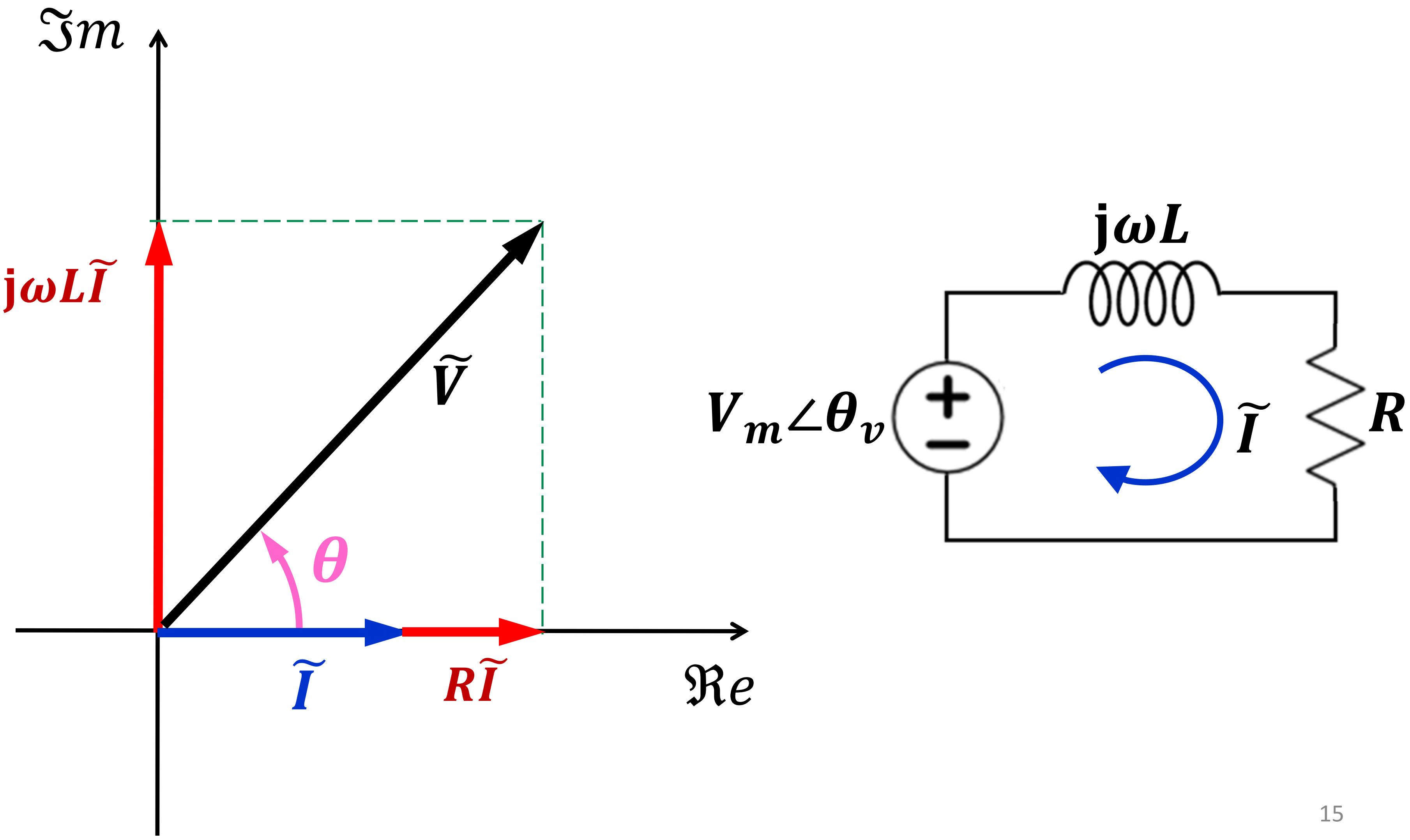
Current LAGS voltage by 90°
(it reaches peak value later)



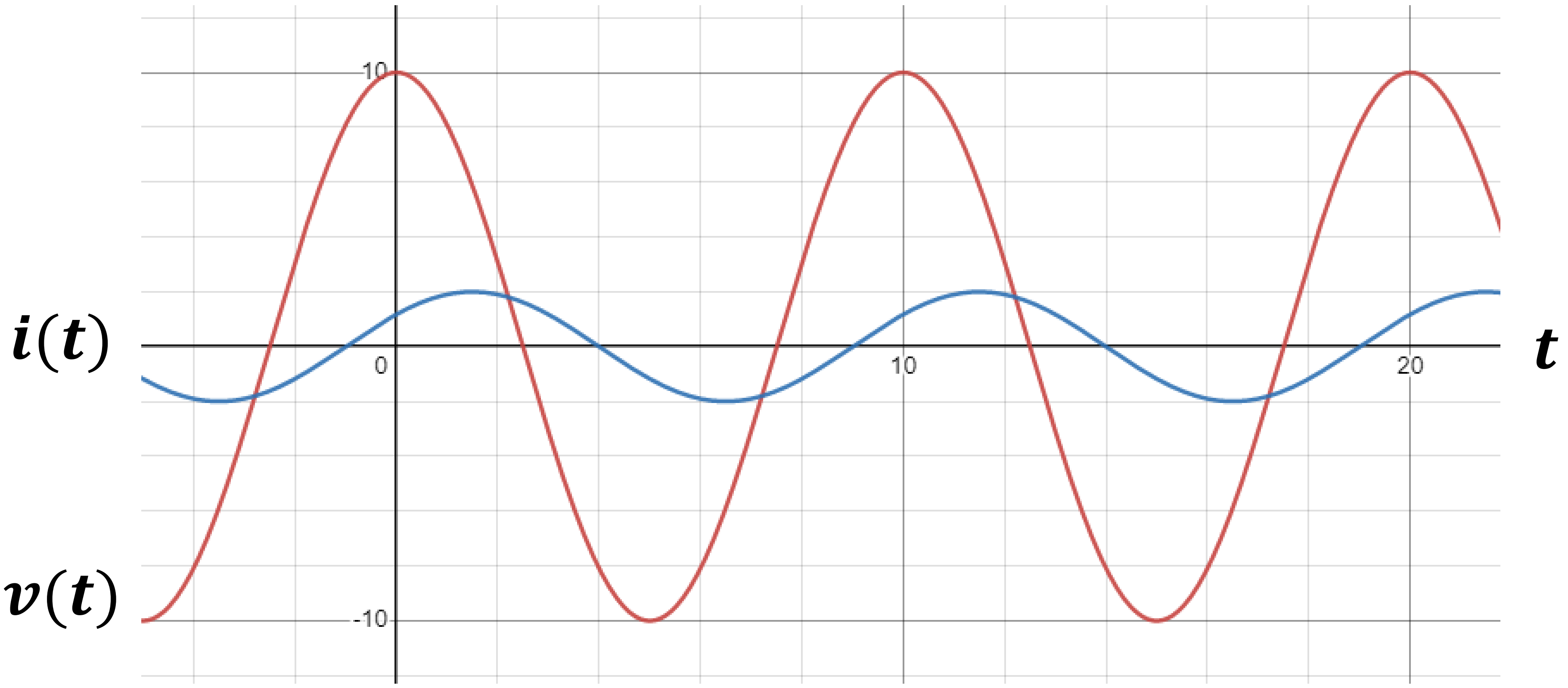
Waveforms are in "quadrature"

Total reactance is inductive: current LAGS voltage

$$\tilde{V} = Z\,\tilde{I} = R\tilde{I} + j\omega L\,\tilde{I}$$



Current LAGS voltage
(it reaches peak value later)



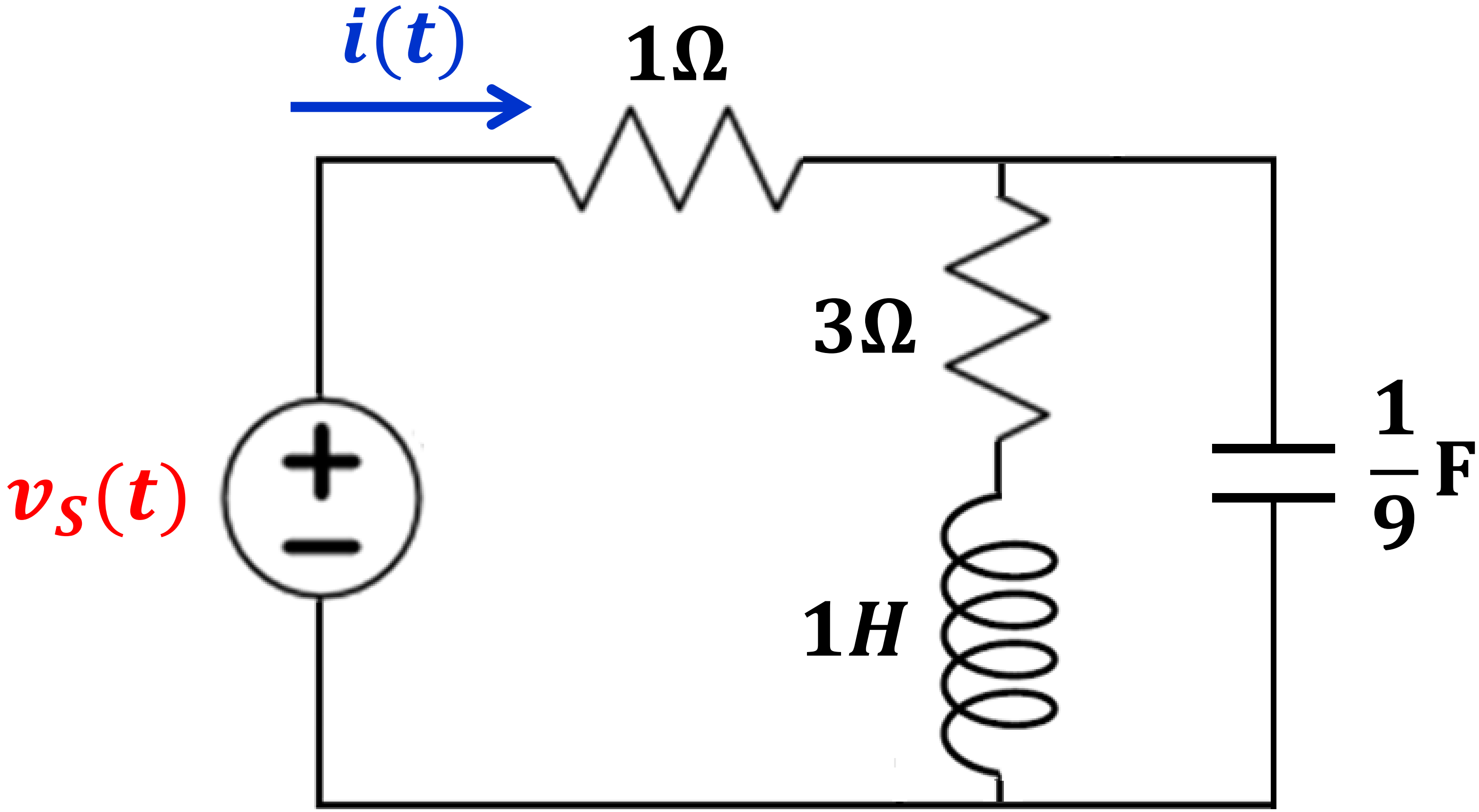
But waveforms are NOT in "quadrature"

Example of Phasor Analysis

$v_S(t) = 10\cos(3t)$

$\omega = 3\ \text{rad/s}$

Determine $i(t)$



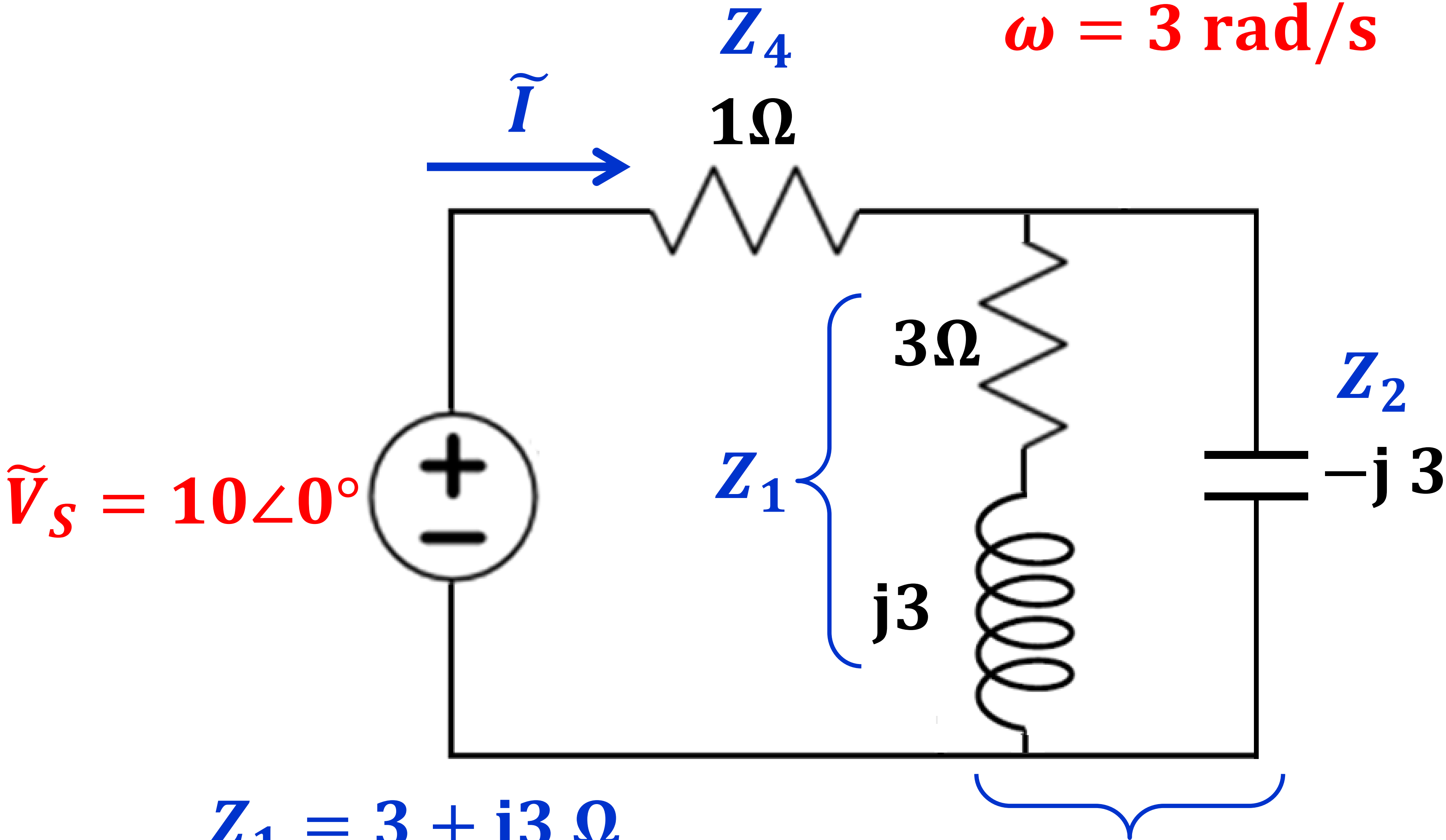
$$j\omega L = j \times 3 \times 1H = j3\ \Omega$$

$$1/j\omega C = -j/\omega C = -j/(3 \times 1/9\ \text{F}) = -j3\ \Omega$$

Phasor form

$v_S(t) = 10\cos(3t)$

$\omega = 3$ rad/s



$Z_1 = 3 + j3\ \Omega$

$Z_2 = -j3\ \Omega$

$Z_3 = Z_1 // Z_2$

$$Z_1 = 3 + j3\ \Omega \qquad Z_2 = -j3\ \Omega$$

$$Z_3 = Z_1//Z_2 = \left(\frac{1}{3+j3} + \frac{1}{-j3}\right)^{-1} =$$

$$= \left(\frac{\cancel{-j3} + 3 + \cancel{j3}}{-j3(3+j3)}\right)^{-1} = \left(\frac{3}{9-j9}\right)^{-1} = (3 - j3)\Omega$$

$$Z_1 = 3 + j3\ \Omega \qquad Z_2 = -j3\ \Omega$$

$$Z_3 = Z_1//Z_2 = \left(\frac{1}{3+j3} + \frac{1}{-j3}\right)^{-1} =$$

$$= \left(\frac{\cancel{-j3} + 3 + \cancel{j3}}{-j3(3+j3)}\right)^{-1} = \left(\frac{3}{9-j9}\right)^{-1} = (3-j3)\Omega$$

$$Z_{eq} = \underset{1\Omega}{Z_4} + \underset{(3-j3)\Omega}{Z_1//Z_2} = 1 + 3 - j3 = (4-j3)\Omega$$

$$\tilde{I} = \frac{\tilde{V}_S}{Z_{eq}} = \frac{10\angle 0^\circ}{(4-j3)}\ \mathrm{A}$$

$$\boxed{\tilde{I} = \frac{\tilde{V}_S}{Z_{eq}} = \frac{10\angle 0^\circ}{(4 - j3)}\ \text{A}}$$

$$\tilde{I} = \frac{10(4 + j3)}{(4 - j3)(4 + j3)} = \frac{40 + j30}{16 + 9 - \cancel{j12} + \cancel{j12}}$$

$$\tilde{I} = \frac{40 + j30}{25} = 1.6 + j\,1.2\ \text{A}$$

$$\tilde{I} = \frac{\tilde{V}_S}{Z_{eq}} = \frac{10\angle 0°}{(4 - j3)} \text{ A}$$

$$\tilde{I} = \frac{10(4 + j3)}{(4 - j3)(4 + j3)} = \frac{40 + j30}{16 + 9 - \cancel{j12} + \cancel{j12}}$$

$$\tilde{I} = \frac{40 + j30}{25} = 1.6 + j\,1.2 \text{ A}$$

$$|\,\tilde{I}\,| = \sqrt{1.6^2 + 1.2^2} = \sqrt{4} = 2 \text{ A}$$

$$\angle \tilde{I} = \theta_I = \tan^{-1}\left(\frac{1.2}{1.6}\right) = 0.6435 \text{ rad} = 36.87°$$

$$\tilde{I} = \frac{\tilde{V}_S}{Z_{eq}} = \frac{10\angle 0°}{(4 - \mathrm{j}3)}\ \mathrm{A}$$

Alternative approach

$$|\ \tilde{I}\ | = \frac{|10\angle 0°|}{|4 - j3|} = \frac{10}{\sqrt{4^2 + 3^2}} = \frac{10}{\sqrt{25}} = \frac{10}{5} = 2\ \mathrm{A}$$

$$\angle\tilde{I} = \theta_I = 0 - \tan^{-1}\left(\frac{-3}{4}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\angle\tilde{I} = 0.6435\ \mathrm{rad} = 36.87°$$

$$\tilde{I} = \frac{\tilde{V}_S}{Z_{eq}} = \frac{10\angle 0°}{(4 - j3)} \text{ A}$$

$$|\tilde{I}| = 2 \text{ A}$$

$$\angle\tilde{I} = 0.6435 \text{ rad} = 36.87°$$

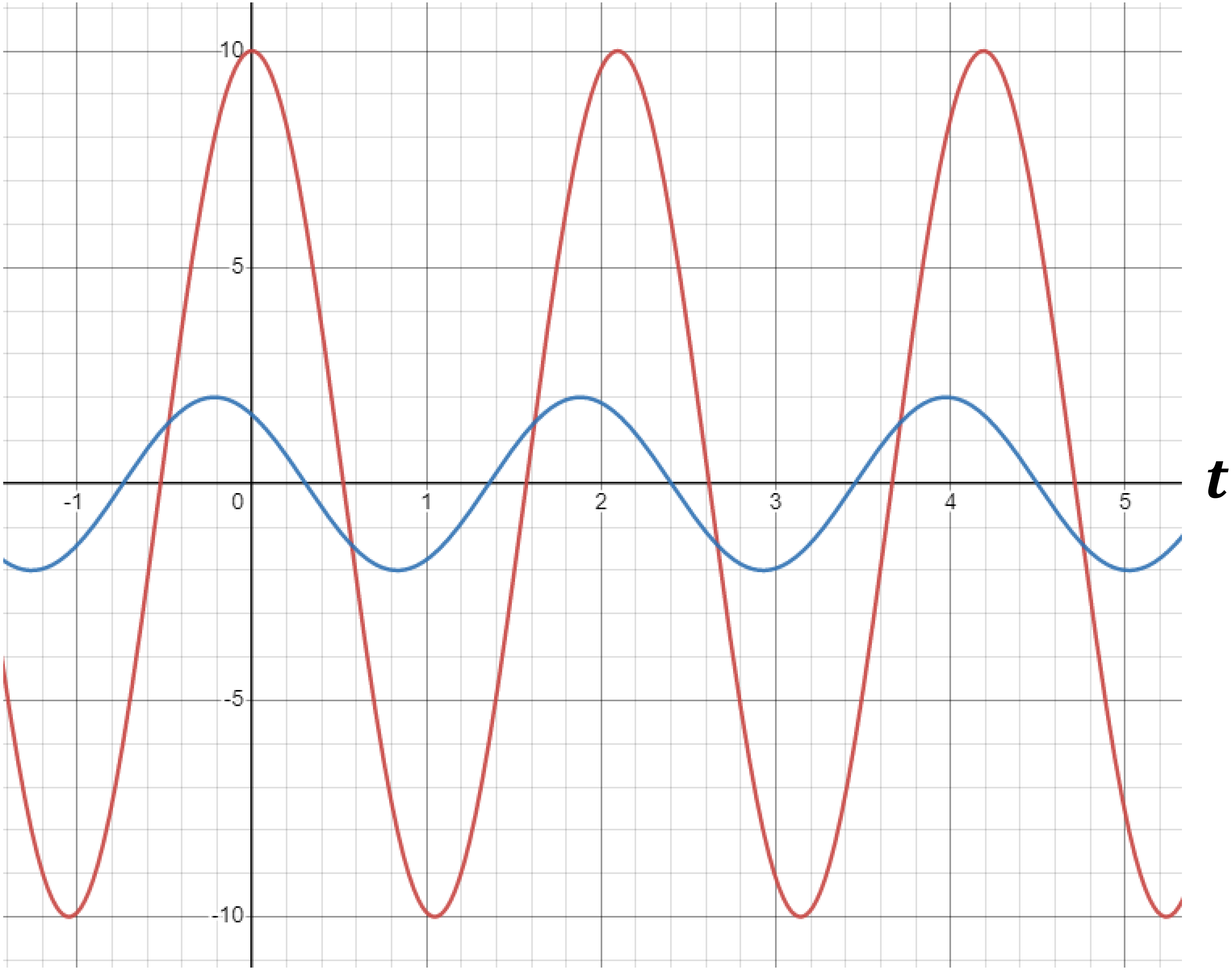
$$\tilde{I} = 2\angle 36.87° \text{ A}$$

or

$$\tilde{I} = 2\exp(j36.87°) \text{ A}$$

$$\Leftrightarrow \quad i(t) = 2\cos(3t + 36.87°) \text{ A}$$

$$v_S = 10\cos(3t)$$



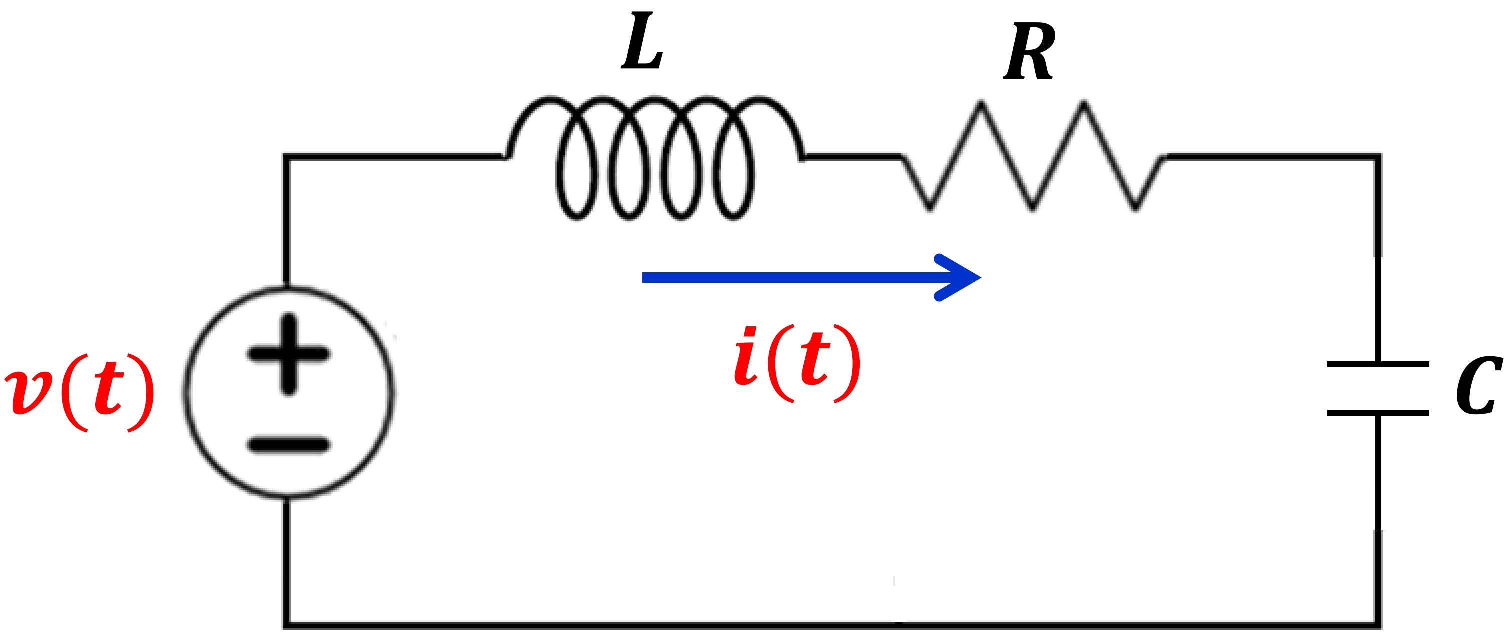
$$i(t) = 2\cos(3t + 36.87°)\ \mathrm{A}$$

EXAMPLES

Resonance in RLC circuits

Let’s take this opportunity also to retrace the steps leading to phasors, for review.

Time-differential equations for time harmonic signals can be transformed into algebraic equations for phasors.



This RLC circuit is described by the integro-differential equation (KVL)

$$v(t) = L\frac{di(t)}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

$$\int_{-\infty}^{t} i(t)dt = Q$$

Represents the total charge accumulated in the capacitor

$$\frac{1}{C}\int_{-\infty}^{t} i(t)dt = \frac{Q}{C} = V_C\ (t)$$

Represents the potential at the terminals of the capacitor

Integro-differential equations are harder to solve, but we can take the derivative

$$v(t) = L\frac{di(t)}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

$$\Downarrow$$

$$\frac{dv(t)}{dt} = L\frac{d^2i(t)}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t)$$

For a time-harmonic excitation, voltage and current will have the form

$$v(t) = V_0 \cos(\omega t + \theta_V)$$

$$i(t) = I_0 \cos(\omega t + \theta_I)$$

with phasors

$$\widetilde{V} = V_0 \exp(\,\mathrm{j}\theta_V)$$

$$\widetilde{I} = I_0 \exp(\,\mathrm{j}\theta_I)$$

If the voltage excitation is given:

$V_0 \quad \theta_V$ **are inputs**

$I_0 \quad \theta_I$ **are unknowns**

Time-differential equations for time harmonic signals can be transformed into algebraic equations for phasors.

Time domain

$$\frac{dv(t)}{dt} = L\frac{d^2 i(t)}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t)$$

Phasor Transformation

Phasor (frequency) domain

$$\mathrm{j}\omega\,\tilde{V} = L\,\mathrm{j}\omega\left(\mathrm{j}\omega\,\tilde{I}\right) + R\,\mathrm{j}\omega\tilde{I} + \frac{1}{C}\,\tilde{I}$$

$$\tilde{V} = \mathrm{j}\omega L\,\tilde{I} + R\,\tilde{I} + \frac{1}{\mathrm{j}\omega C}\,\tilde{I}$$

$$\tilde{V} = \underbrace{\left(R + \mathrm{j}\omega L - \mathrm{j}\frac{1}{\omega C}\right)}_{\text{Impedance } Z}\tilde{I}$$

$$\tilde{V} = \left(R + \mathrm{j}\omega L - \mathrm{j}\frac{1}{\omega C} \right) \tilde{I} = Z\,\tilde{I}$$

is a new form of Ohm's law!

Complex		*Real*		*Imaginary*
Z	$=$	R	$+$	$\mathrm{j}\left(\omega L - \frac{1}{\omega C} \right)$
Impedance		**Resistance**		**Reactance**

The equation is easily solved

$$\tilde{V} = Z\,\tilde{I} \quad \rightarrow \quad \tilde{I} = \frac{\tilde{V}}{Z}$$

$$\tilde{I} = \frac{\tilde{V}}{\left(R + \mathrm{j}\omega L - \mathrm{j}\frac{1}{\omega C}\right)} = I_0 \exp(\mathrm{j}\theta_I)$$

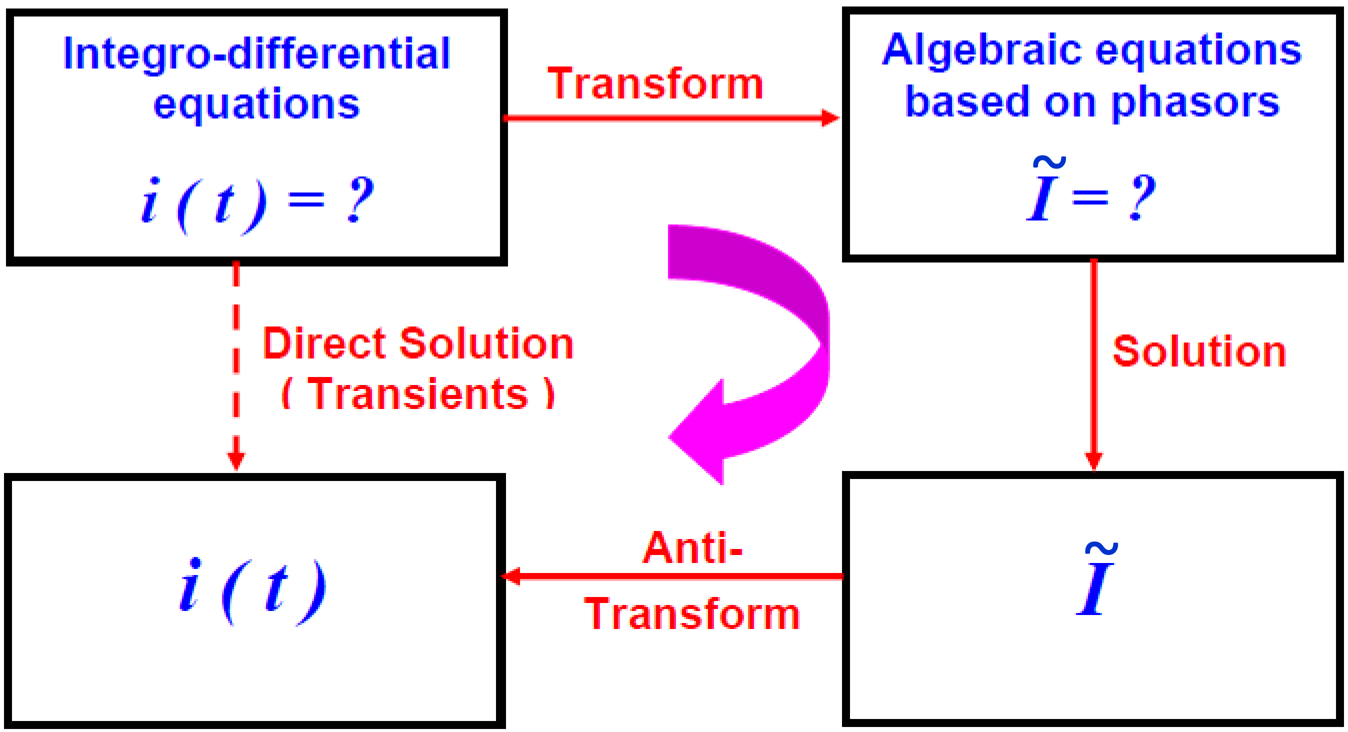
to obtain the unknowns I_0 and θ_I

The time-dependent solution is obtained from a backward phasor transformation (anti-transform)

$$i(t) = \Re e\{I_0 \exp(\mathrm{j}\theta_I)\exp(\mathrm{j}\omega \mathrm{t})\}$$
$$= I_0 \cos(\omega t + \theta_I)$$

The phasor formalism has provided a convenient way to solve time-harmonic problems in steady state, without differential equations (which are only needed for transients).

The exponential representation of phasors allows immediate separation of frequency and phase information.



The simple RLC circuit example with elements in series illustrates clearly the main properties of an impedance

- **The resistance is not function of ω**
- **The inductive component is proportional to ω**
- **The capacitive component is inversely proportional to ω**

Inductive components are positive

Capacitive components are negative

In series connection, inductive and capacitive components simply add up.

Resonance

At a certain frequency ω_r the magnitudes of the two reactive terms are equal, so they cancel out (<u>together, they behave like a short circuit</u>)

→ **the impedance becomes purely resistive.**

$$\omega_r L = \frac{1}{\omega_r C}$$

$$\boxed{\omega_r = \frac{1}{\sqrt{LC}}}$$

resonance condition

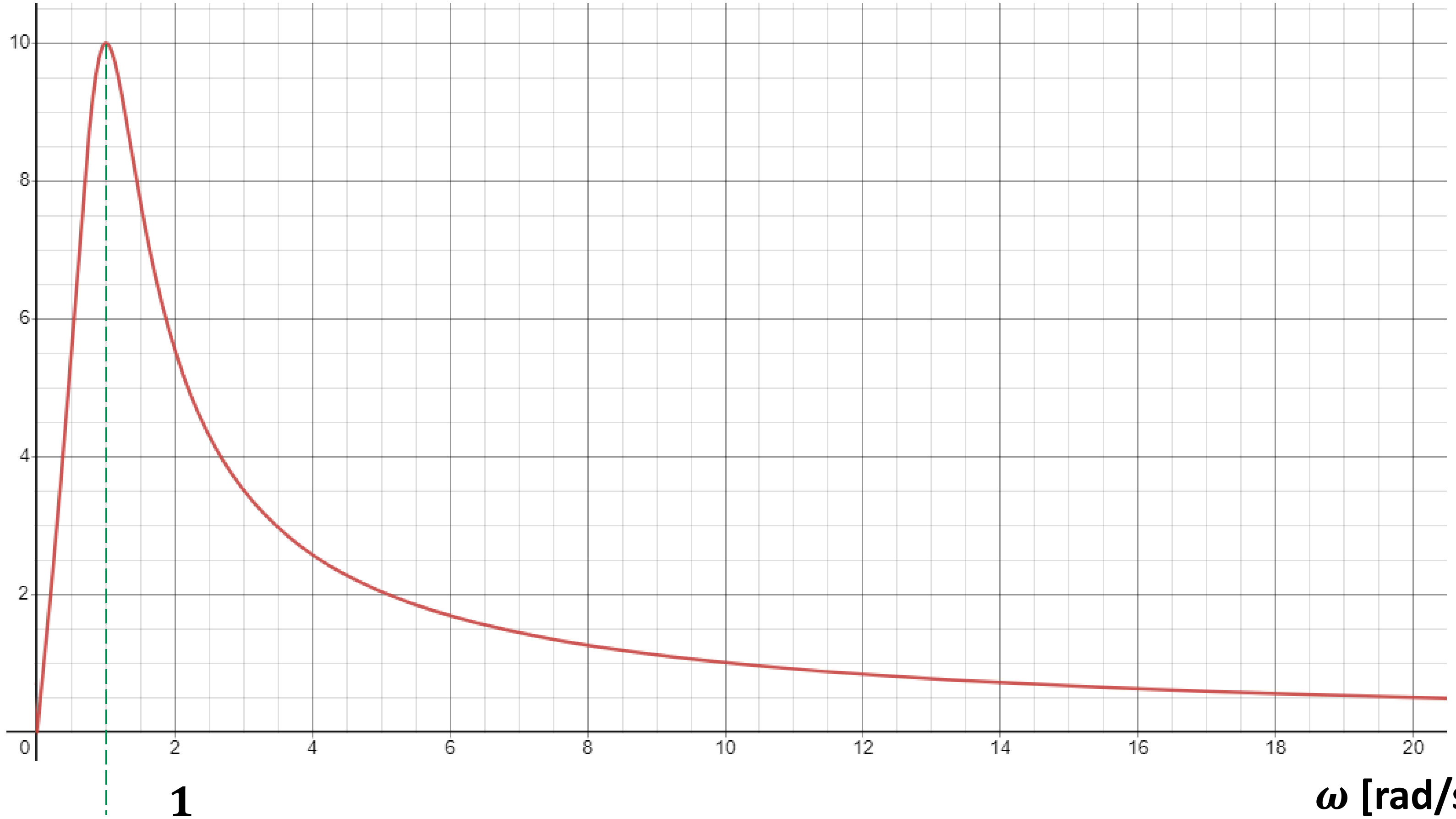
The peak value of the current is maximum at resonance

$$|\tilde{I}| = \frac{|\tilde{V}|}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Example:

$V_0 = 10\text{V}$ $\quad R = 1\Omega$

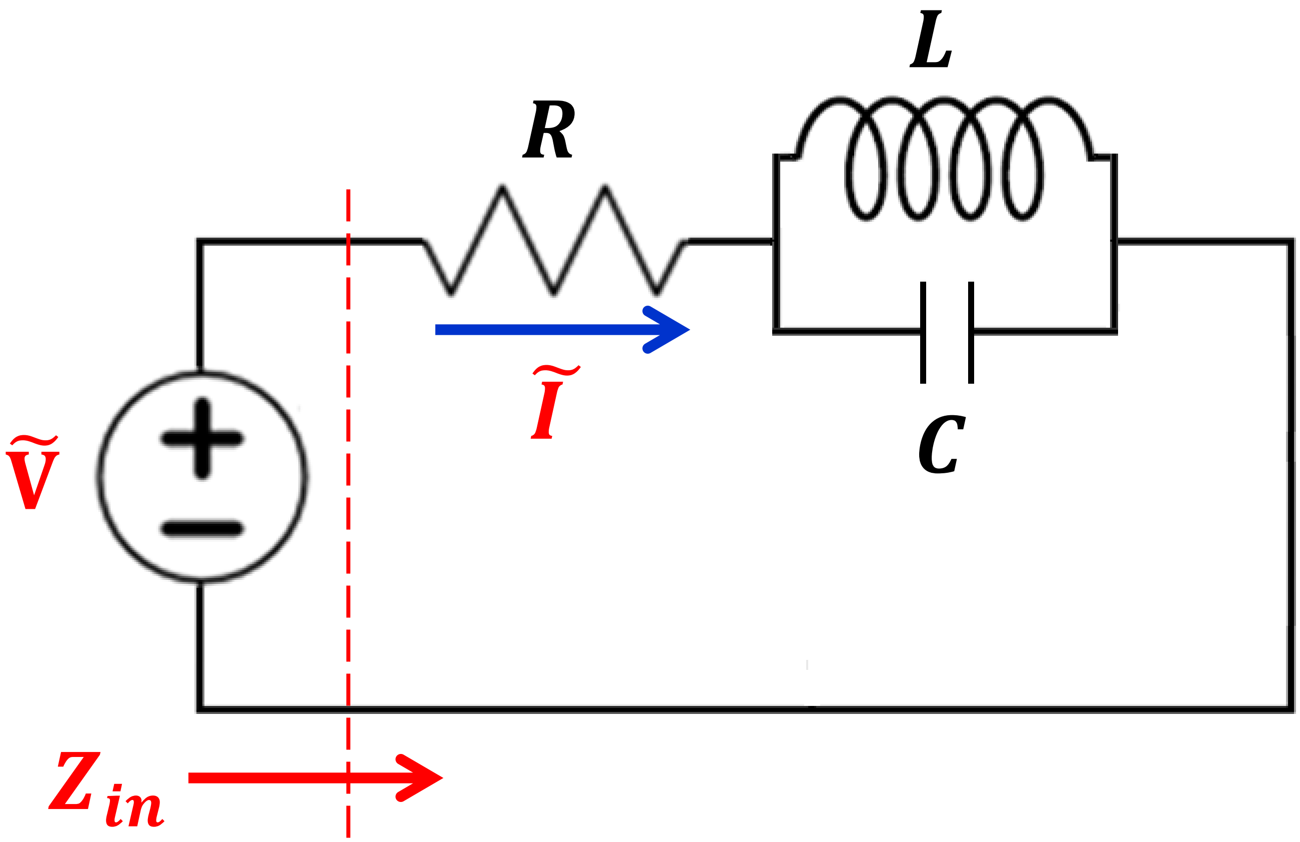
$L = 1\text{H}$ $\quad C = 1\text{F}$



ω [rad/s]

$$\omega_r = \frac{1}{\sqrt{LC}}$$

Consider now a circuit with L and C in parallel



$$Z_{in} = R + \left(\frac{1}{j\omega L} + j\omega C\right)^{-1} = R + \frac{j\omega L}{1 - \omega^2 LC}$$

$$Z_{in} = R + \left(\frac{1}{j\omega L} + j\omega C\right)^{-1} = R + \frac{j\omega L}{1 - \omega^2 LC}$$

When:

$\omega = 0$	$Z_{in} = R$	
$\omega = \frac{1}{\sqrt{LC}}$	$Z_{in} \to \infty$	**resonance condition**
$\omega \to \infty$	$Z_{in} = R$	

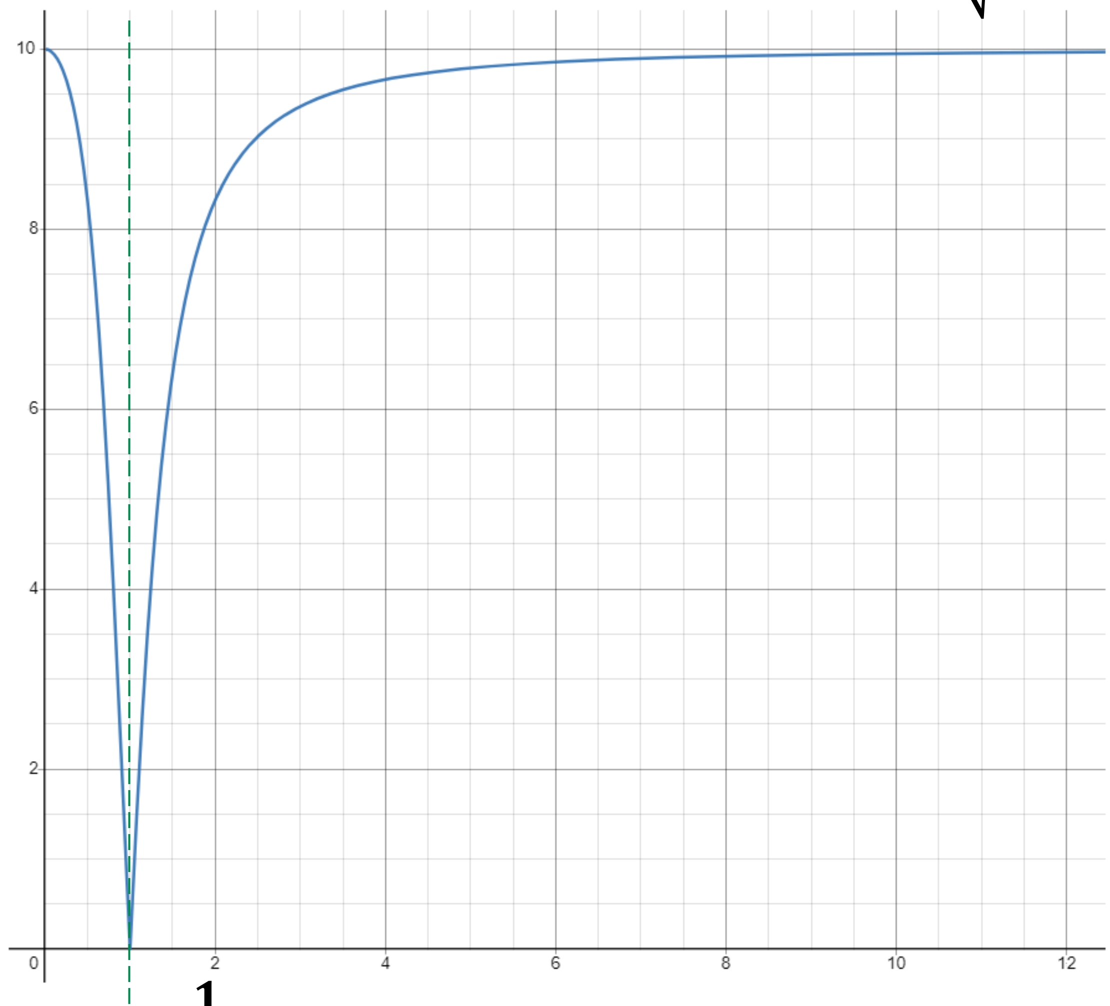
At the resonance condition, the reactance (parallel of L and C) behaves like an open circuit and no current can flow.

The peak value of the current is zero at resonance

$V_0 = 10\text{V}$ $\quad R = 1\Omega$

$L = 1\text{H}$ $\quad C = 1\text{F}$

$$|\tilde{I}| = \frac{|\tilde{V}|}{\sqrt{R^2 + \left(\frac{\omega L}{1-\omega^2 LC}\right)^2}}$$



$$\omega_r = \frac{1}{\sqrt{LC}}$$

ω [rad/s]

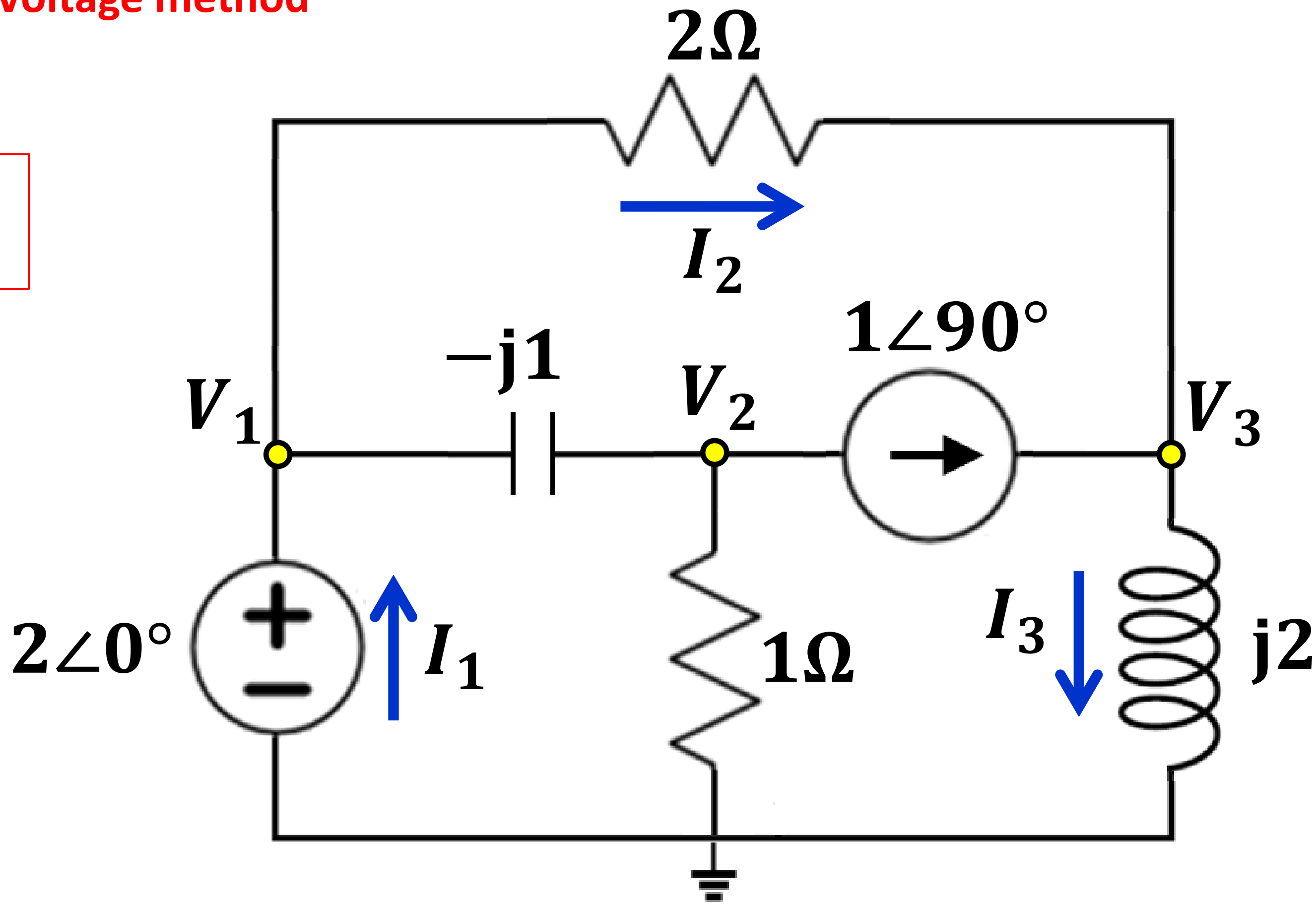
Find phasors $V_1, V_2, V_3, I_1, I_2, I_3$

(We can drop the wavy hat ˜ from now on, since we are getting used to phasors)

Let's use Node Voltage method

By inspection

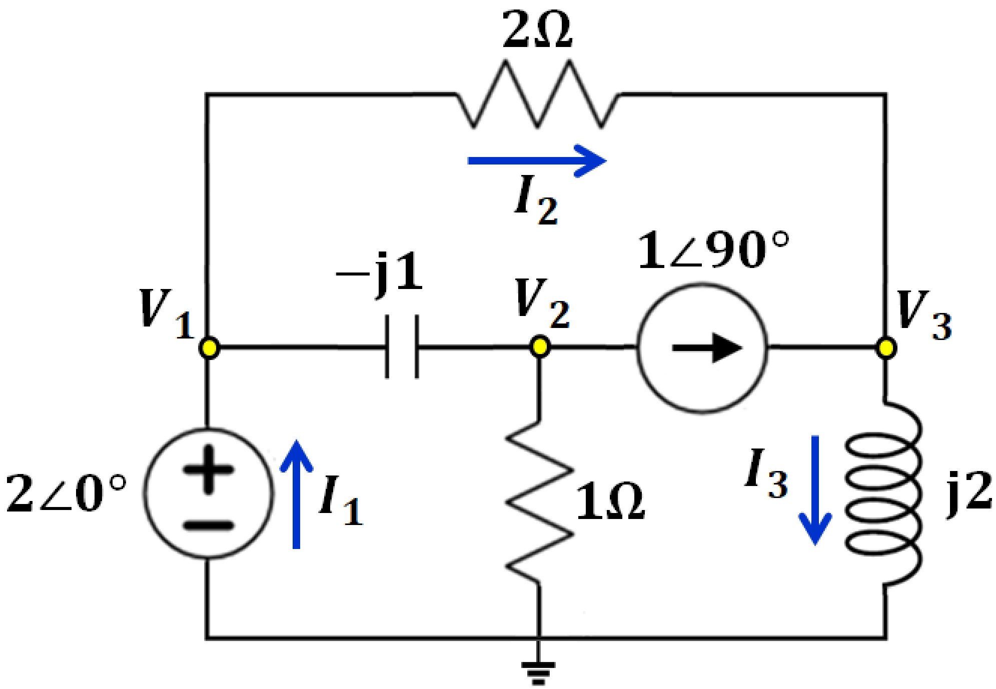
$$V_1 = 2\text{V}$$



Node 2 KCL

$$\frac{V_2 - V_1}{-\text{j}1} + \frac{V_2}{1} + 1\angle 90° = 0$$

$$V_1 = 2\text{V}$$



Node 2 KCL

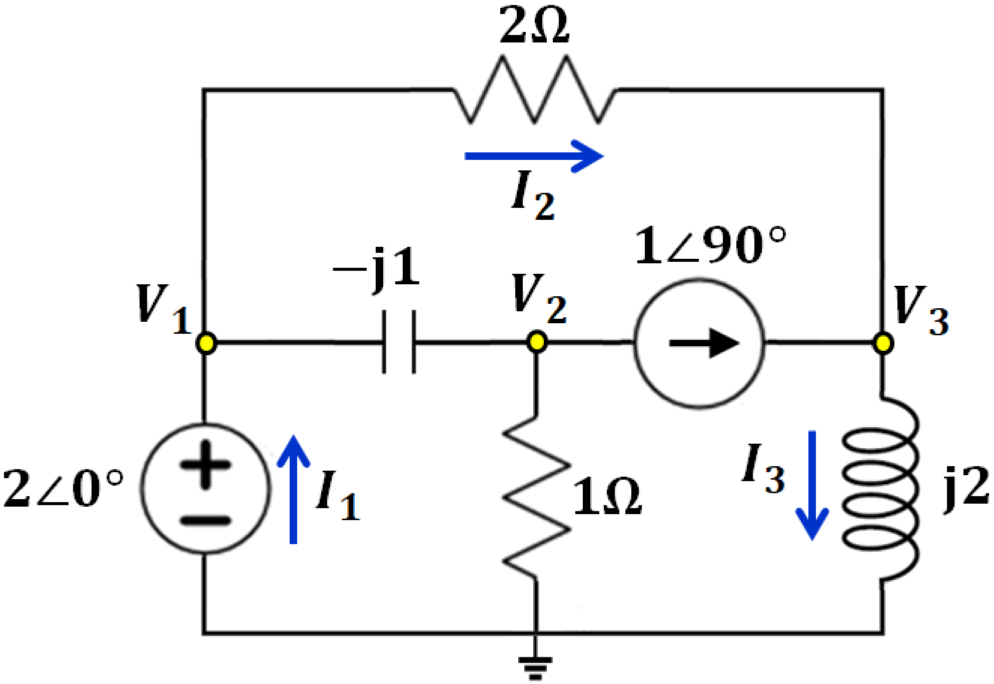
$$\frac{V_2 - V_1}{-\text{j}1} + \frac{V_2}{1} + 1\angle 90° = 0$$

$$\text{j}V_2 - 2\text{j} + V_2 + \text{j}1 = 0 \quad \Rightarrow \quad V_2(1+\text{j}) = \text{j}1$$

$$V_2 = \frac{\text{j}1}{(1+\text{j})} = \frac{\text{j}1(1-\text{j})}{(1+\text{j})(1-\text{j})} = \frac{1+\text{j}}{2} = 0.5 + \text{j}0.5$$

$$V_1 = 2\text{V}$$

$$V_2 = 0.5 + j0.5\ \text{V}$$



Node 3 KCL

$$\frac{V_3}{j2} + \frac{V_3 - 2}{2} - 1\angle 90° = 0$$

$$-j\frac{1}{2}V_3 + \frac{1}{2}V_3 - 1 - j1 = 0 \Rightarrow V_3 = \frac{1 + j1}{(1 - j1)/2}$$

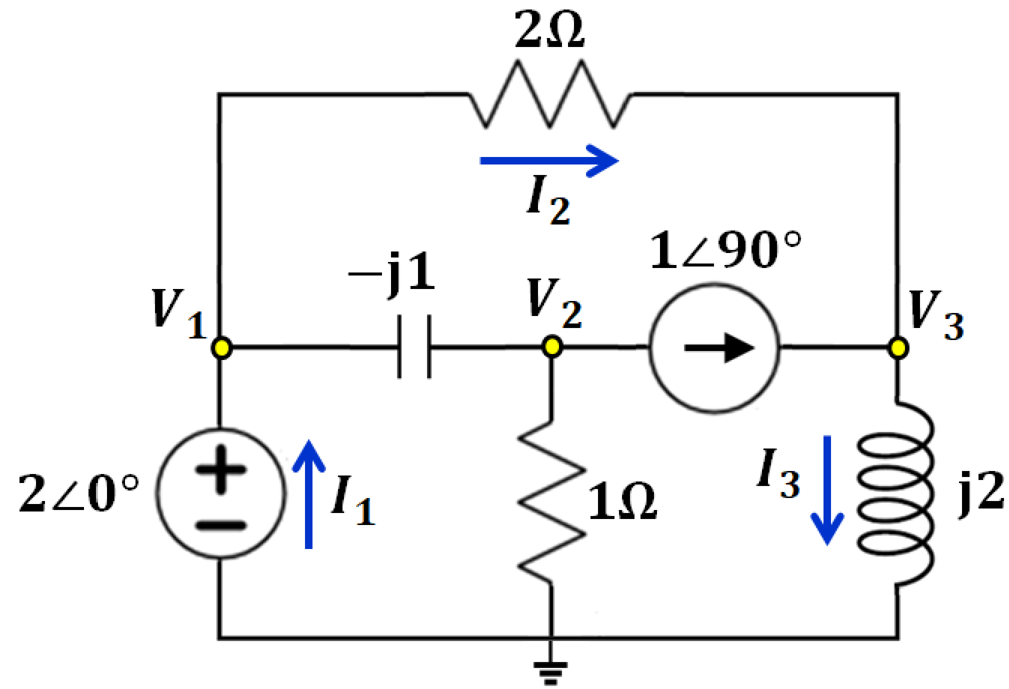
$$V_3 = \frac{\cancel{2}(1 + j1)(1 + j1)}{\cancel{2}} = j2\ \text{V}$$

$V_1 = 2\text{V}$

$V_2 = 0.5 + j0.5\ \text{V}$

$V_3 = j2\text{V}$

Currents



$$I_1 - I_2 = \frac{V_1 - V_2}{-j1} = \frac{2 - (0.5 + j0.5)}{-j1} = 0.5 + j1.5\text{A}$$

$$I_2 = \frac{V_1 - V_2}{2} = \frac{2 - j2}{2} = 1 - j$$

$$I_1 = 1.5 + j0.5\ \text{A}$$

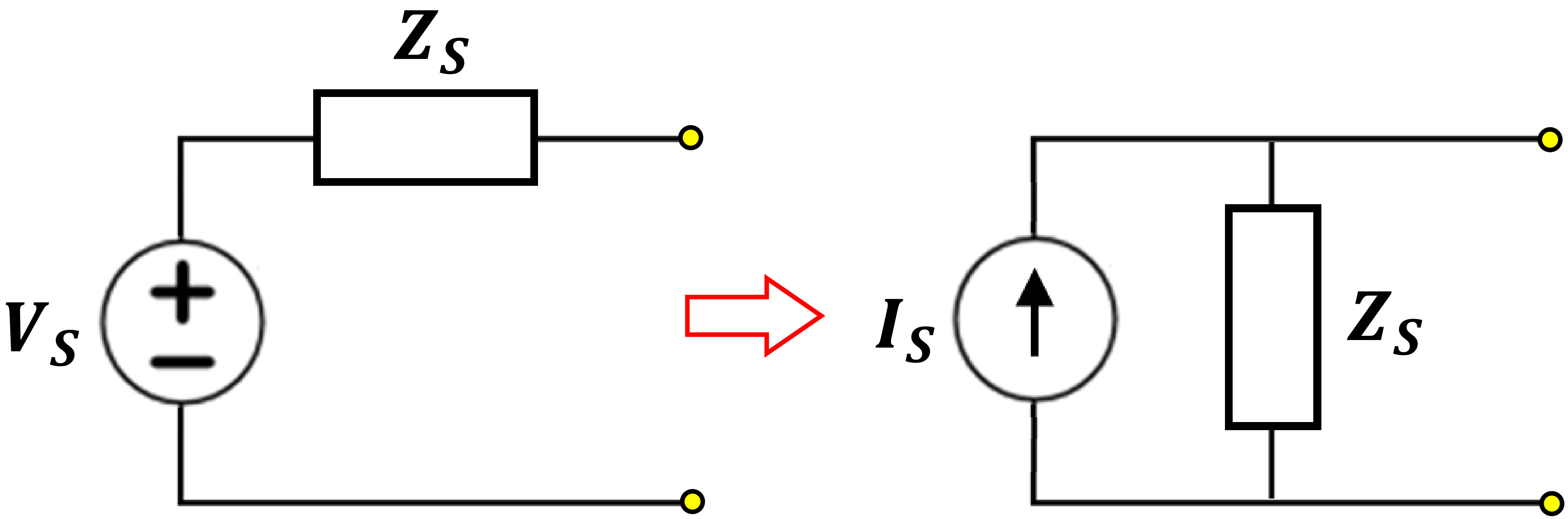
$$I_3 - I_2 - 1\angle 90° = 0 \qquad I_3 - (1 - j1) - j1 = 0$$

Also: $I_3 = V_3/j2 = j2/j2 = 1$

$$I_3 = 1\ \text{A}$$

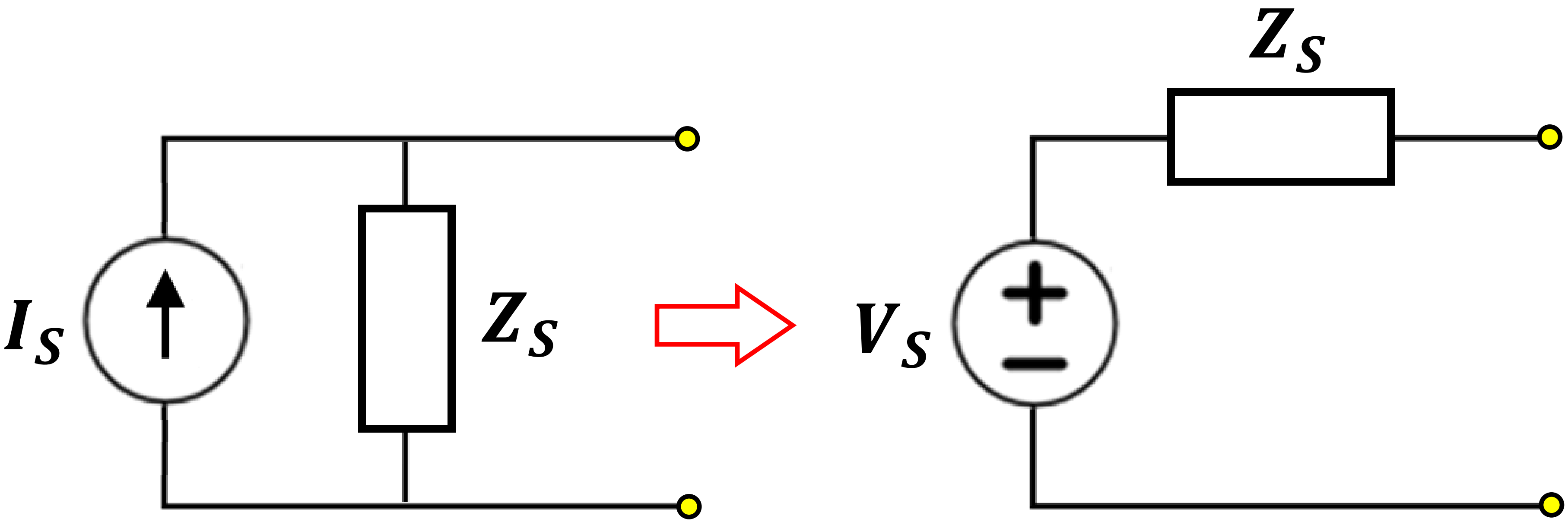
Source Transformations

The approach we used before works for phasors, too.



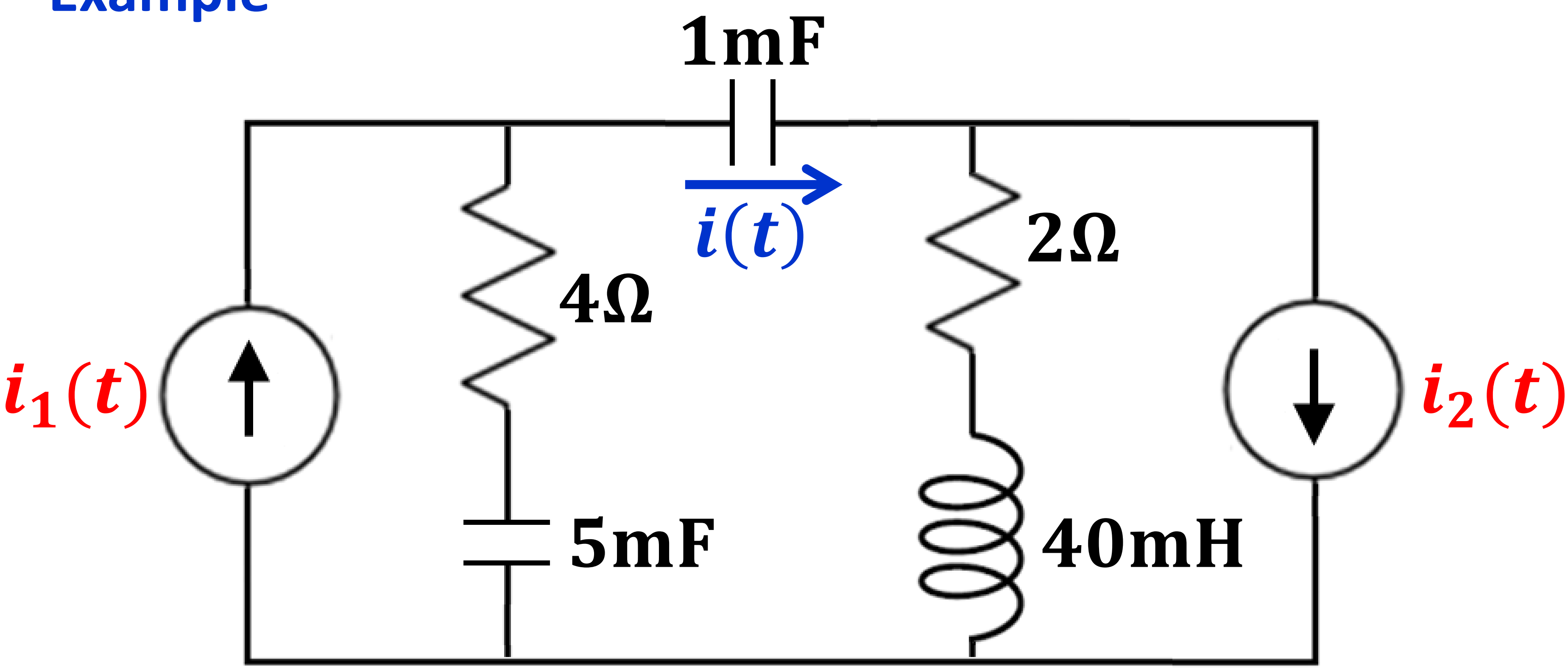
$$I_S = \frac{V_S}{Z_s}$$

Source Transformations



$$V_S = I_S\, Z_s$$

Example



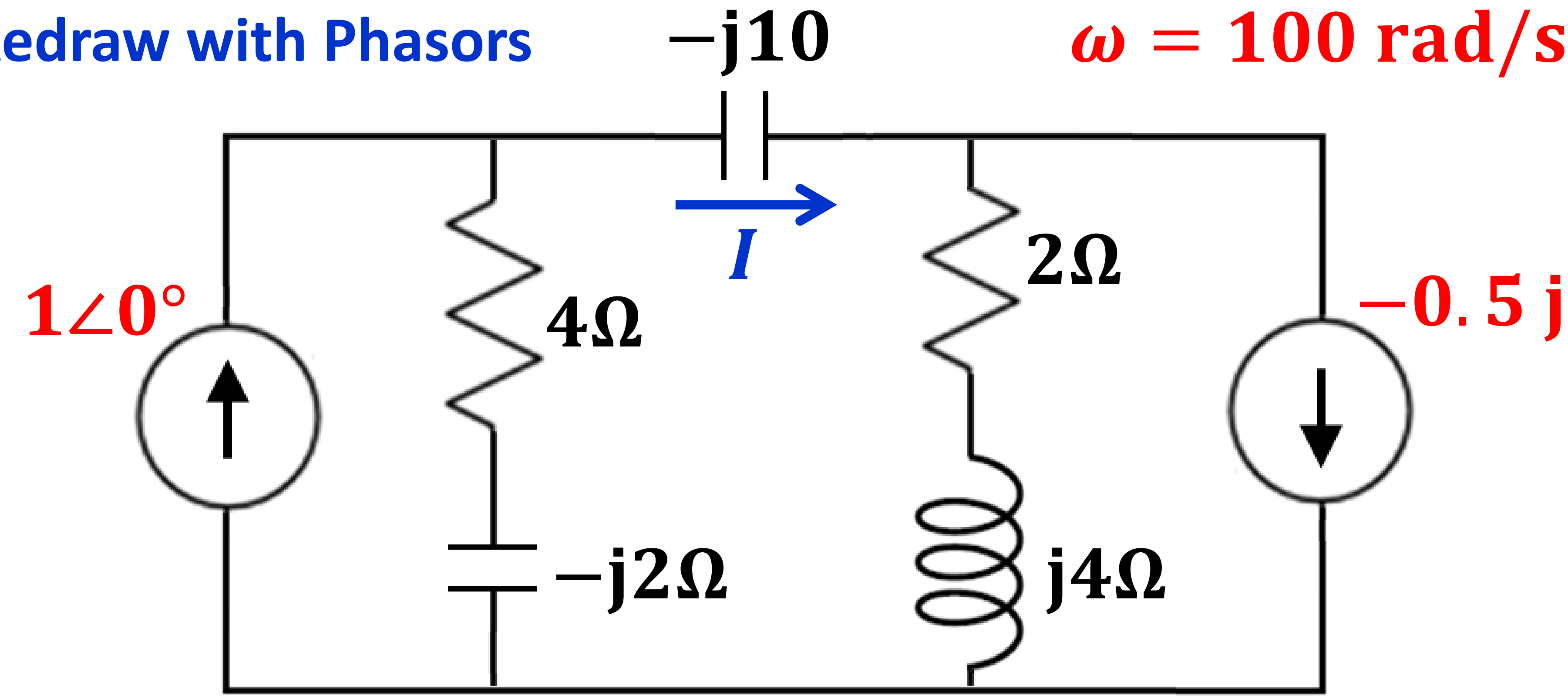
$$i_1(t) = 1\cos(100t)\ \text{A}$$

$$i_2(t) = 0.5\cos(100t - 90°)\ \text{A}$$

Find $i(t)$

Redraw with Phasors

$\omega = 100\ \mathrm{rad/s}$

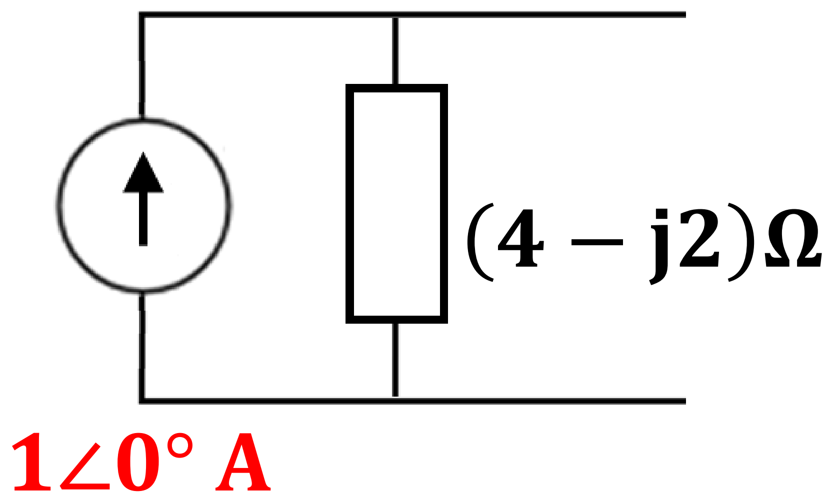


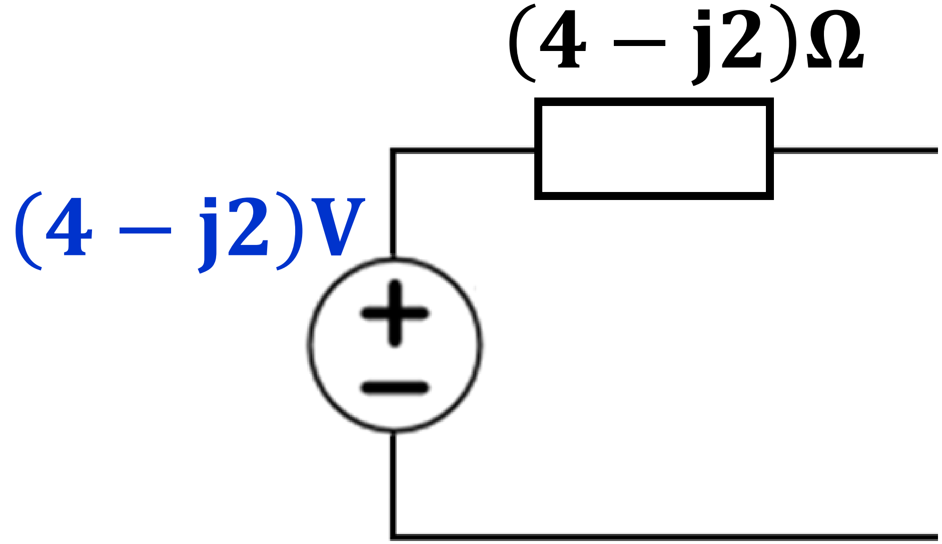
$$C_1 = 5\mathrm{mF} \rightarrow \quad -\mathrm{j}\frac{1}{\omega C_1} = -\mathrm{j}\frac{1}{100 \times 5 \times 10^{-3}} = -\mathrm{j}2\Omega$$

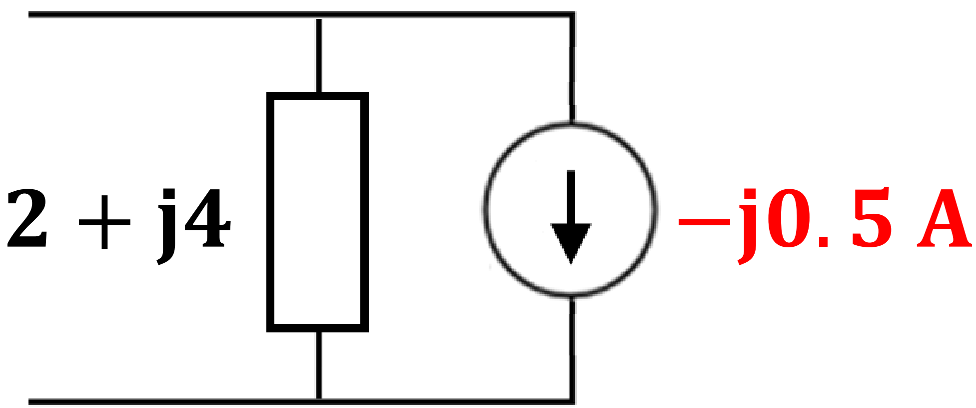
$$C_2 = 1\mathrm{mF} \rightarrow \quad -\mathrm{j}\frac{1}{\omega C_1} = -\mathrm{j}\frac{1}{100 \times 10^{-3}} = -\mathrm{j}10\Omega$$

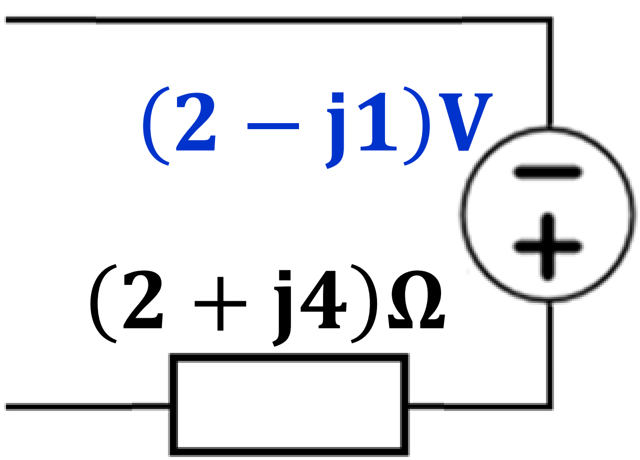
$$L_1 = 40\mathrm{mH} \rightarrow \quad \mathrm{j}\omega L_1 = \mathrm{j}100 \times 40 \times 10^{-3} = \mathrm{j}4\ \Omega$$

Source Transformations

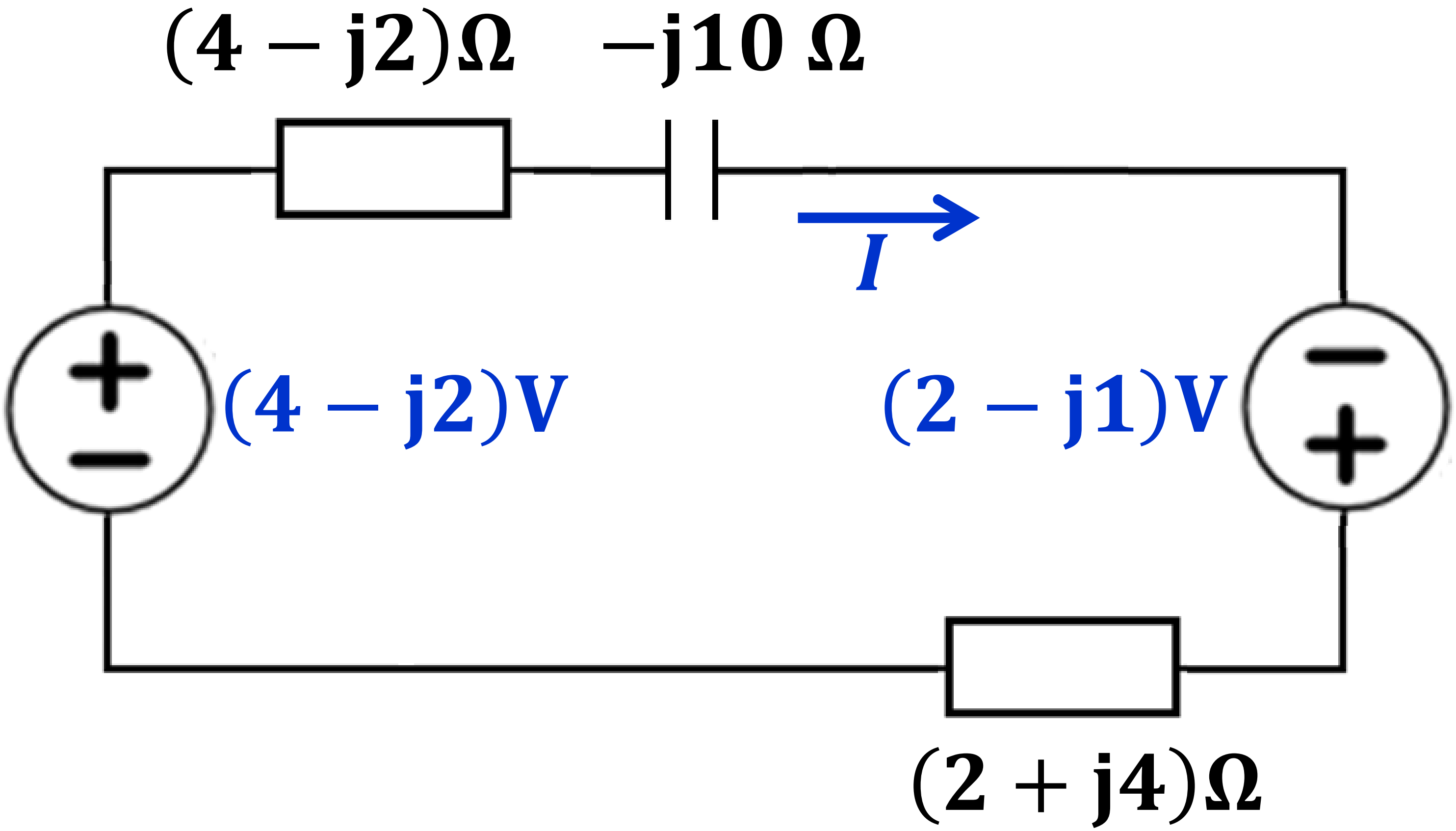








$$V_T = -j0.5(2 + j4) = (2 - j1)\ \text{V}$$

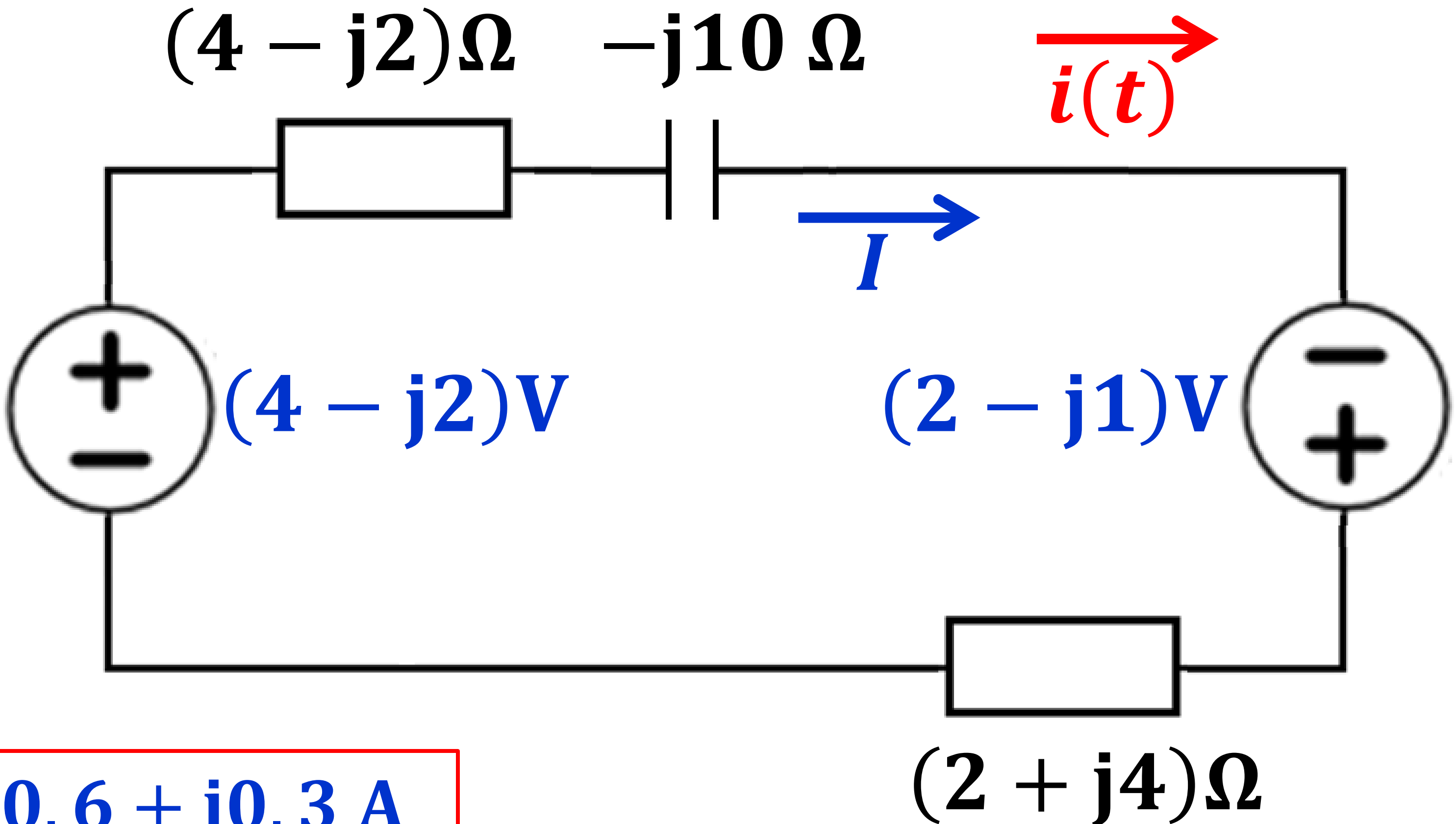


KVL

$$-(4-\mathrm{j}2)-(2-\mathrm{j}1)+I[(4-\mathrm{j}2)-\mathrm{j}10+(2+\mathrm{j}4)]=0$$

$$\Rightarrow \quad (6-\mathrm{j}3)=I(6-\mathrm{j}8)$$

$$I=\frac{6-\mathrm{j}3}{6-\mathrm{j}8}=\frac{(6-\mathrm{j}3)(6+\mathrm{j}8)}{100}=\frac{60+\mathrm{j}30}{100}=0.6+\mathrm{j}0.3\ \mathrm{A}$$



$$I = 0.6 + j0.3\ \text{A}$$

$$I = \sqrt{0.6^2 + 0.3^2}\,\exp\left(j\tan^{-1}\left(\frac{0.3}{0.6}\right)\right)$$

$$= 0.6708\angle 0.46365\text{rad} = 0.6708\angle 26.57°$$

$$\Rightarrow \quad i(t) = 0.6708\cos(100t + 26.57°)$$