ECE 205 “Electrical and Electronics Circuits”

Spring 2024 – LECTURE 21

MWF – 12:00pm

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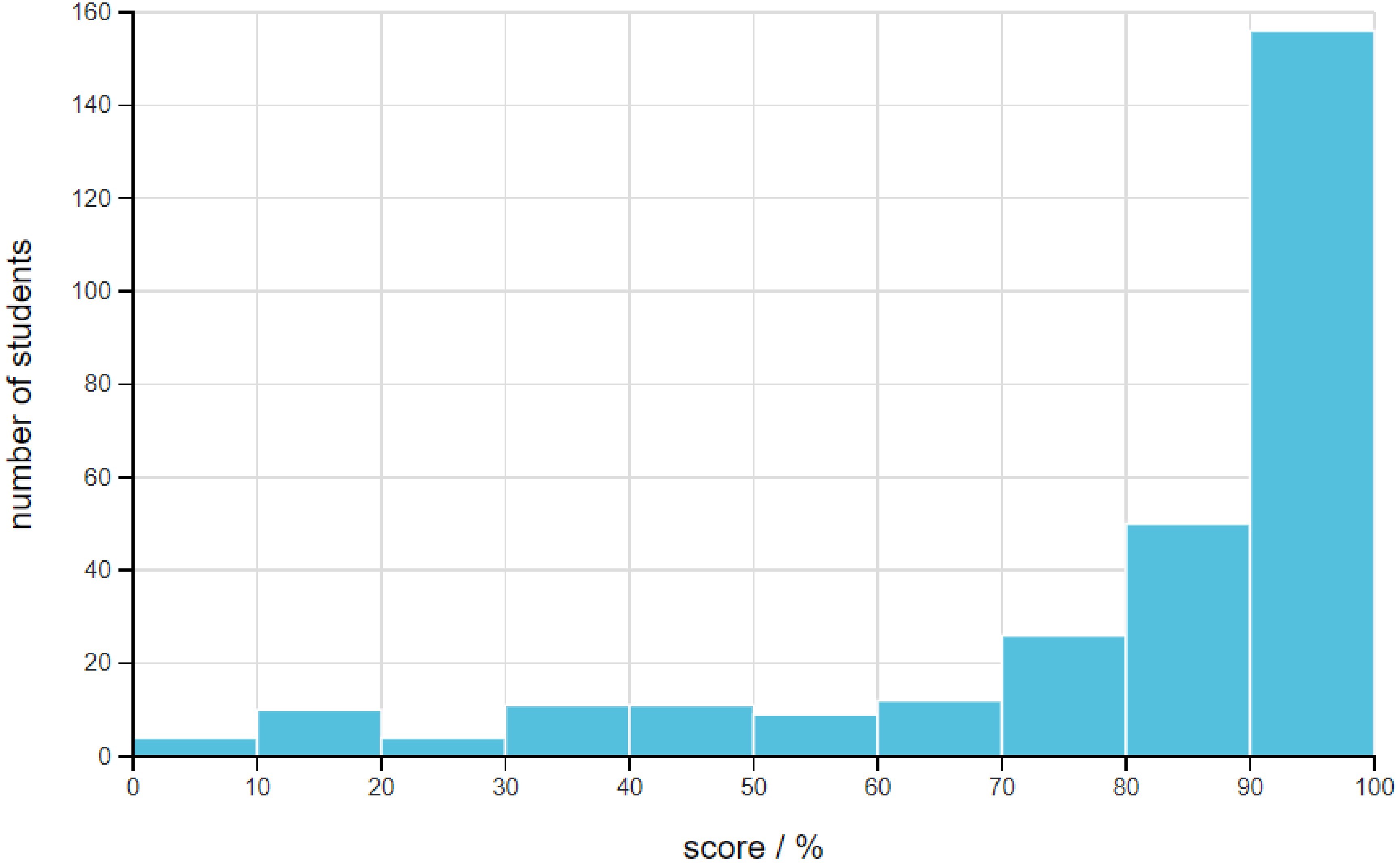
2062 ECE Building

Lectures 21 – Summary

Learning Objectives

1. **Solution of circuit problems with phasors**

Quiz 2 – Grade Distribution



Quiz 2 – Statistics

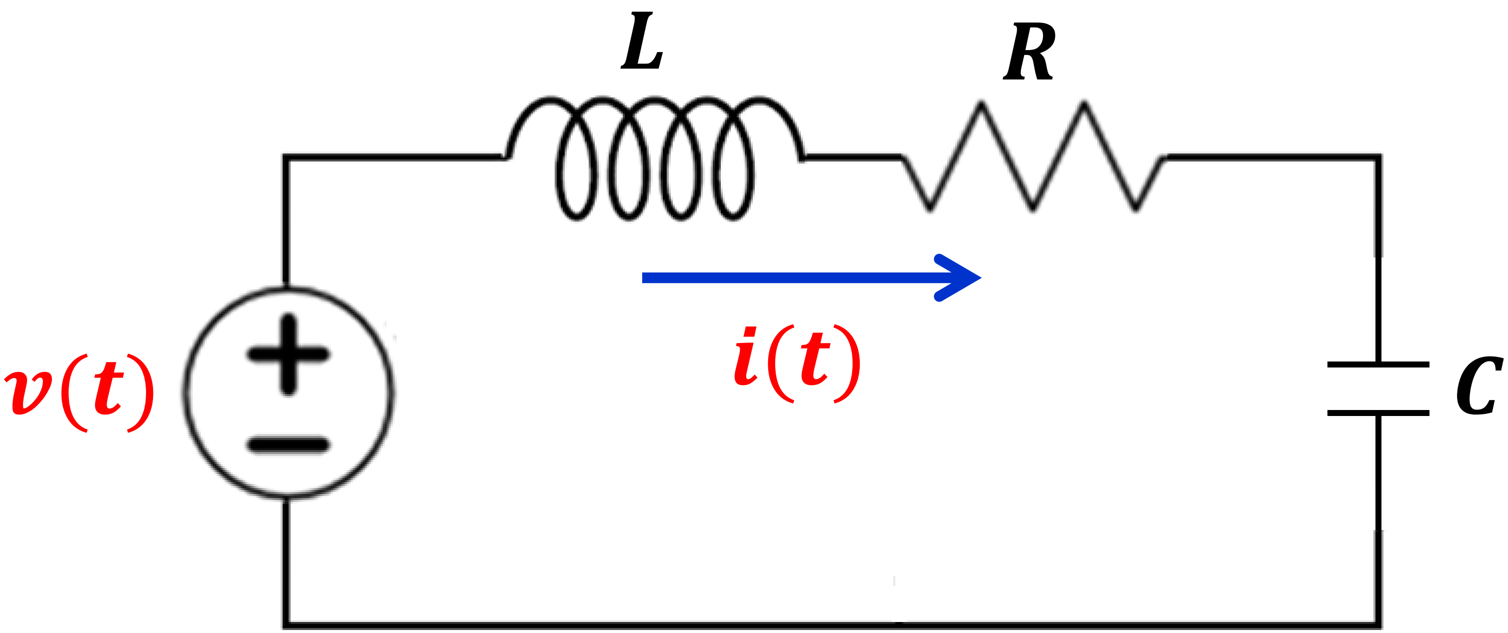
Number of students	293
Mean score	81%
Standard deviation	24%
Median score	91%
Minimum score	2%
Maximum score	100%
Number of 0%	0 (0% of class)
Number of 100%	30 (10% of class)

EXAMPLES

Resonance in RLC circuits

Let's take this opportunity also to retrace the steps leading to phasors, for review.

Time-differential equations for time harmonic signals can be transformed into algebraic equations for phasors.



In the previous lecture we found that this circuit is described by a second order differential equation

$$\frac{dv(t)}{dt} = L\frac{d^2 i(t)}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t)$$

Time-differential equations for time harmonic signals can be transformed into algebraic equations for phasors.

Time domain

$$\frac{dv(t)}{dt} = L\frac{d^2 i(t)}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i(t)$$

Phasor Transformation ⇕

Phasor (frequency) domain

$$\mathrm{j}\omega\,\tilde{V} = L\,\mathrm{j}\omega\left(\mathrm{j}\omega\,\tilde{I}\right) + R\,\mathrm{j}\omega\tilde{I} + \frac{1}{C}\,\tilde{I}$$

$$\tilde{V} = \mathrm{j}\omega L\,\tilde{I} + R\,\tilde{I} + \frac{1}{\mathrm{j}\omega C}\,\tilde{I}$$

$$\tilde{V} = \underbrace{\left(R + \mathrm{j}\omega L - \mathrm{j}\frac{1}{\omega C}\right)}_{\text{Impedance } Z}\,\tilde{I}$$

$$\tilde{V} = \left(R + \mathrm{j}\omega L - \mathrm{j}\frac{1}{\omega C} \right) \tilde{I} = Z\,\tilde{I}$$

is a new form of Ohm's law!

$$\underbrace{Z}_{\text{Impedance}} = \underbrace{R}_{\text{Resistance}} + \underbrace{\mathrm{j}\left(\omega L - \frac{1}{\omega C} \right)}_{\text{Reactance}}$$

Complex	Real	Imaginary
Impedance	Resistance	Reactance

The equation is easily solved

$$\widetilde{V} = Z\,\widetilde{I} \qquad \rightarrow \qquad \widetilde{I} = \frac{\widetilde{V}}{Z}$$

$$\widetilde{I} = \frac{\widetilde{V}}{\left(R + \mathrm{j}\omega L - \mathrm{j}\dfrac{1}{\omega C}\right)} = I_0 \exp(\,\mathrm{j}\theta_I)$$

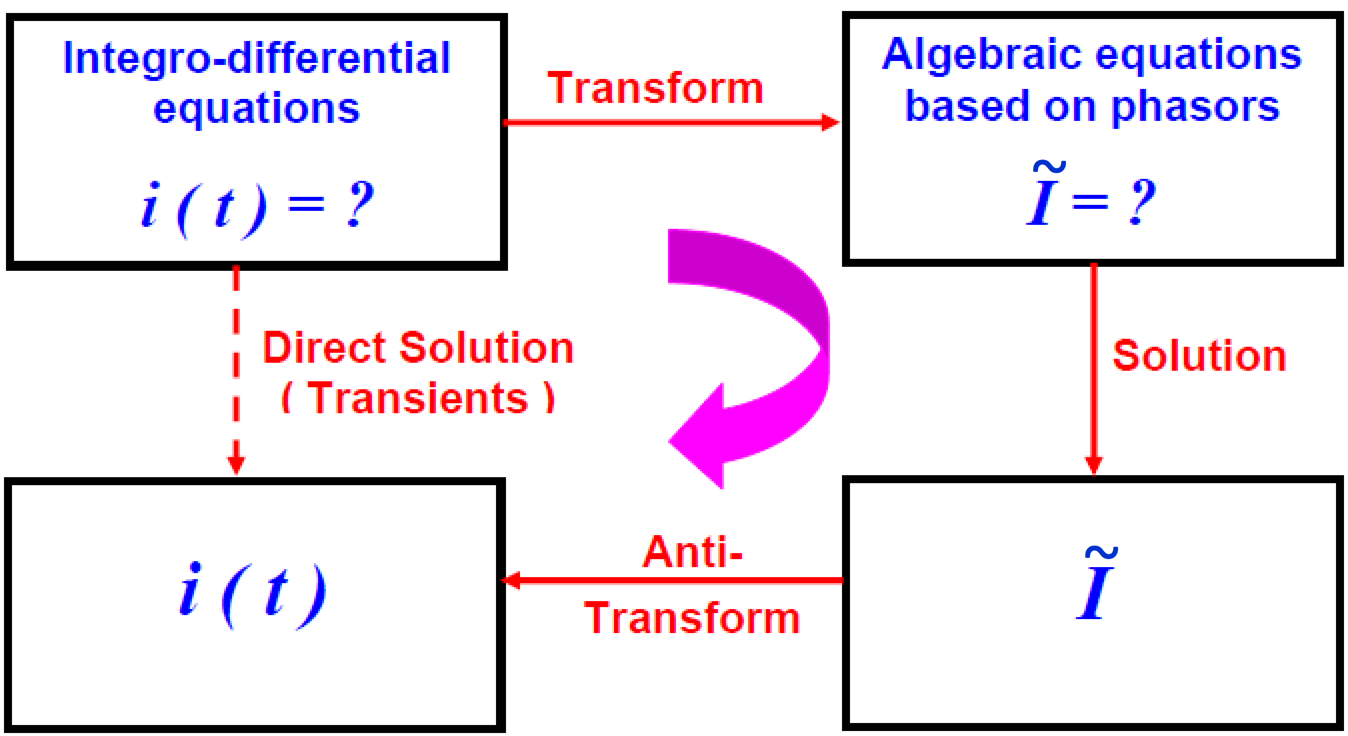
to obtain the unknowns I_0 and θ_I

The time-dependent solution is obtained from a backward phasor transformation (anti-transform)

$$i(t) = \Re e\{I_0 \exp(\,\mathrm{j}\theta_I) \exp(\,\mathrm{j}\omega \mathrm{t})\}$$
$$= I_0 \cos(\omega t + \theta_I)$$

The phasor formalism has provided a convenient way to solve time-harmonic problems in steady state, without differential equations (which are only needed for transients).

The exponential representation of phasors allows immediate separation of frequency and phase information.



The simple RLC circuit example with elements in series illustrates clearly the main properties of an impedance

- **The resistance is not function of ω**
- **The inductive component is proportional to ω**
- **The capacitive component is inversely proportional to ω**

Inductive components are positive

Capacitive components are negative

In series connection, inductive and capacitive components simply add up.

Resonance

At a certain frequency ω_r the magnitudes of the two reactive terms are equal, so they cancel out (<u>together, they behave like a short circuit</u>)

$$Z = R + \mathrm{j}\omega_r L - \mathrm{j}\frac{1}{\omega_r C} = R$$

→ The impedance becomes purely resistive.

$$\omega_r L = \frac{1}{\omega_r C}$$

$$\omega_r = \frac{1}{\sqrt{LC}}$$

resonance condition

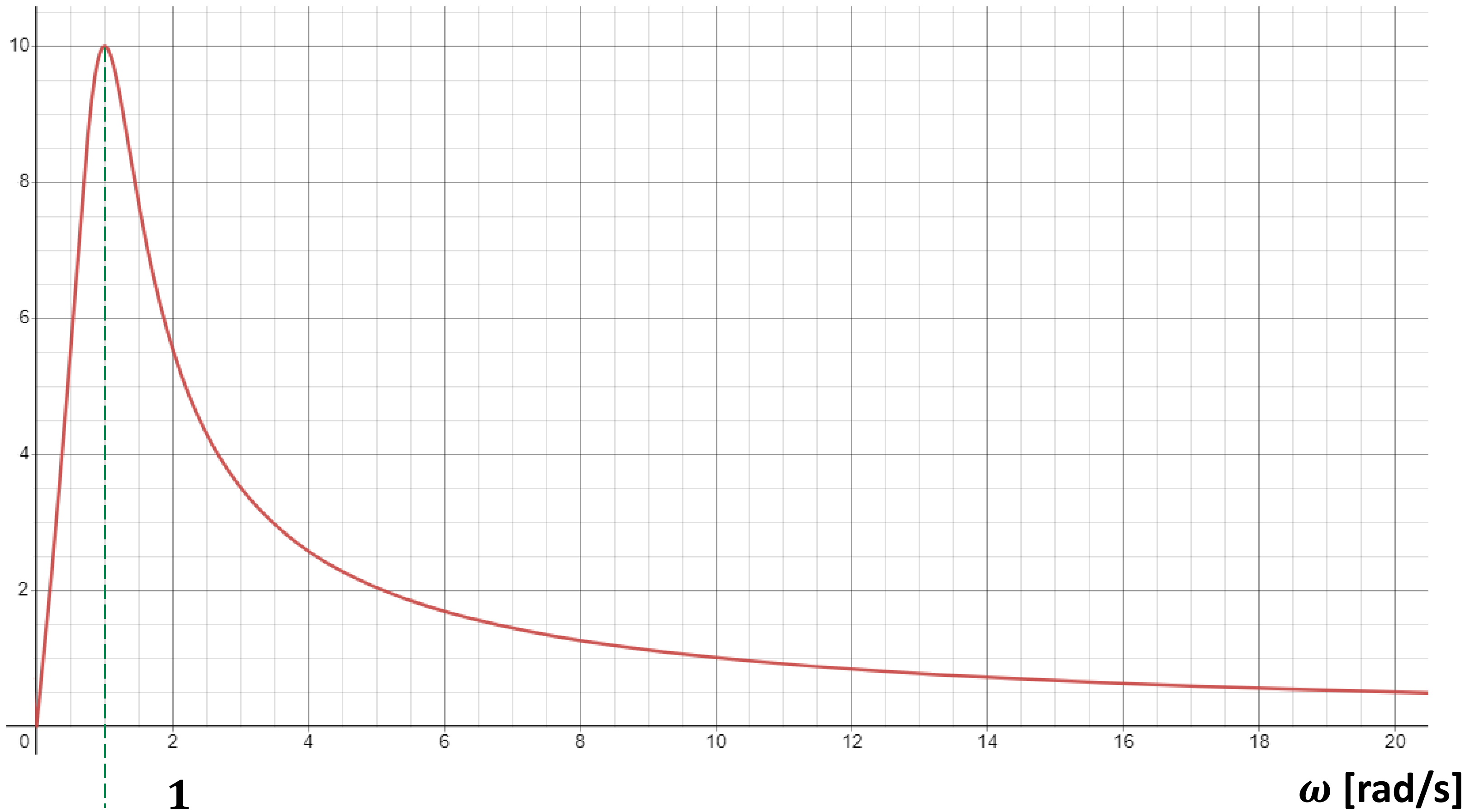
The peak value of the current is maximum at resonance

$$|\tilde{I}| = \frac{|\tilde{V}|}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Example:

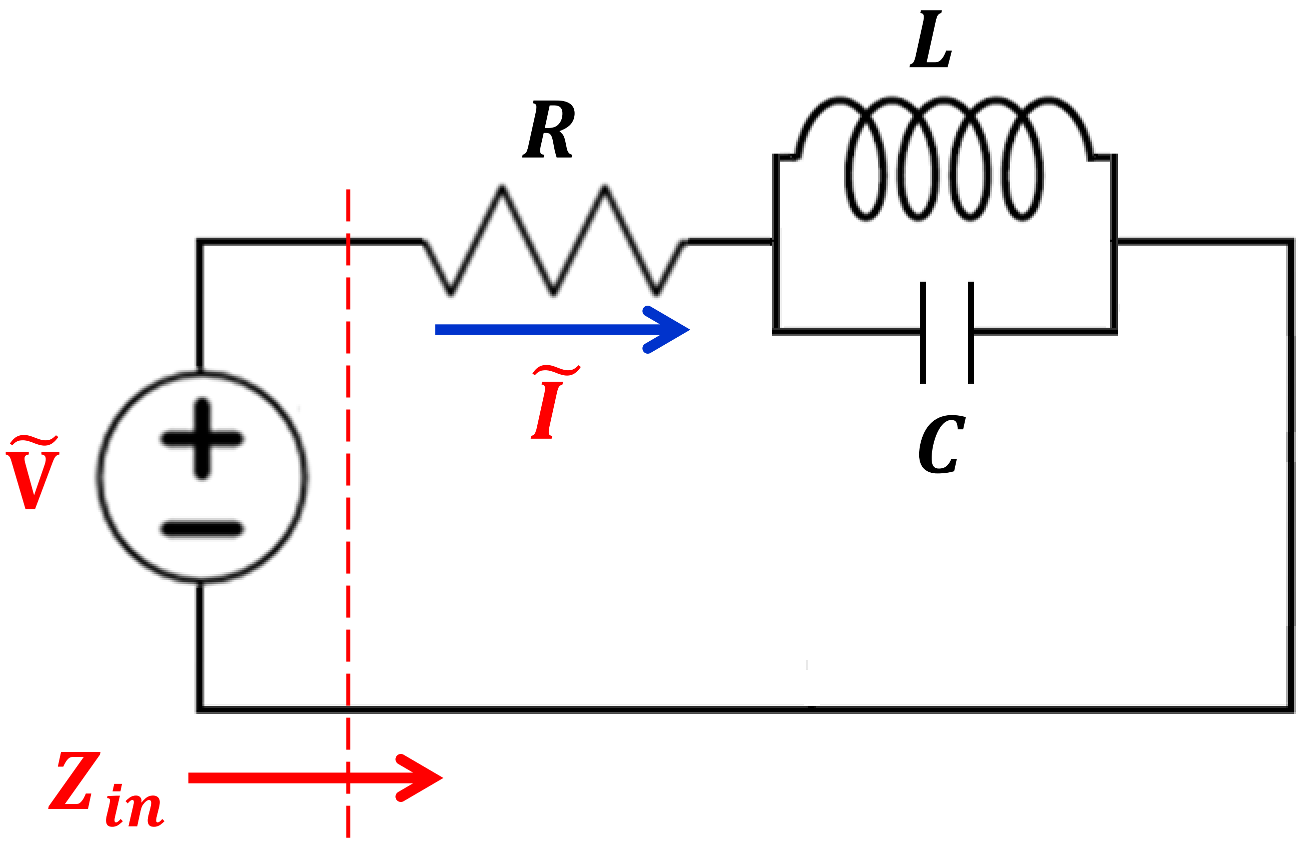
$V_0 = 10\text{V}$ $\quad R = 1\Omega$

$L = 1\text{H}$ $\quad C = 1\text{F}$



$$\omega_r = \frac{1}{\sqrt{LC}}$$

Consider now a circuit with L and C in parallel



$$Z_{in} = R + \left(\frac{1}{j\omega L} + j\omega C\right)^{-1} = R + \frac{j\omega L}{1 - \omega^2 LC}$$

$$Z_{in} = R + \left(\frac{1}{j\omega L} + j\omega C\right)^{-1} = R + \frac{j\omega L}{1 - \omega^2 LC}$$

When:

$\omega = 0$	$Z_{in} = R$	
$\omega = \frac{1}{\sqrt{LC}}$	$Z_{in} \to \infty$	**resonance condition**
$\omega \to \infty$	$Z_{in} = R$	

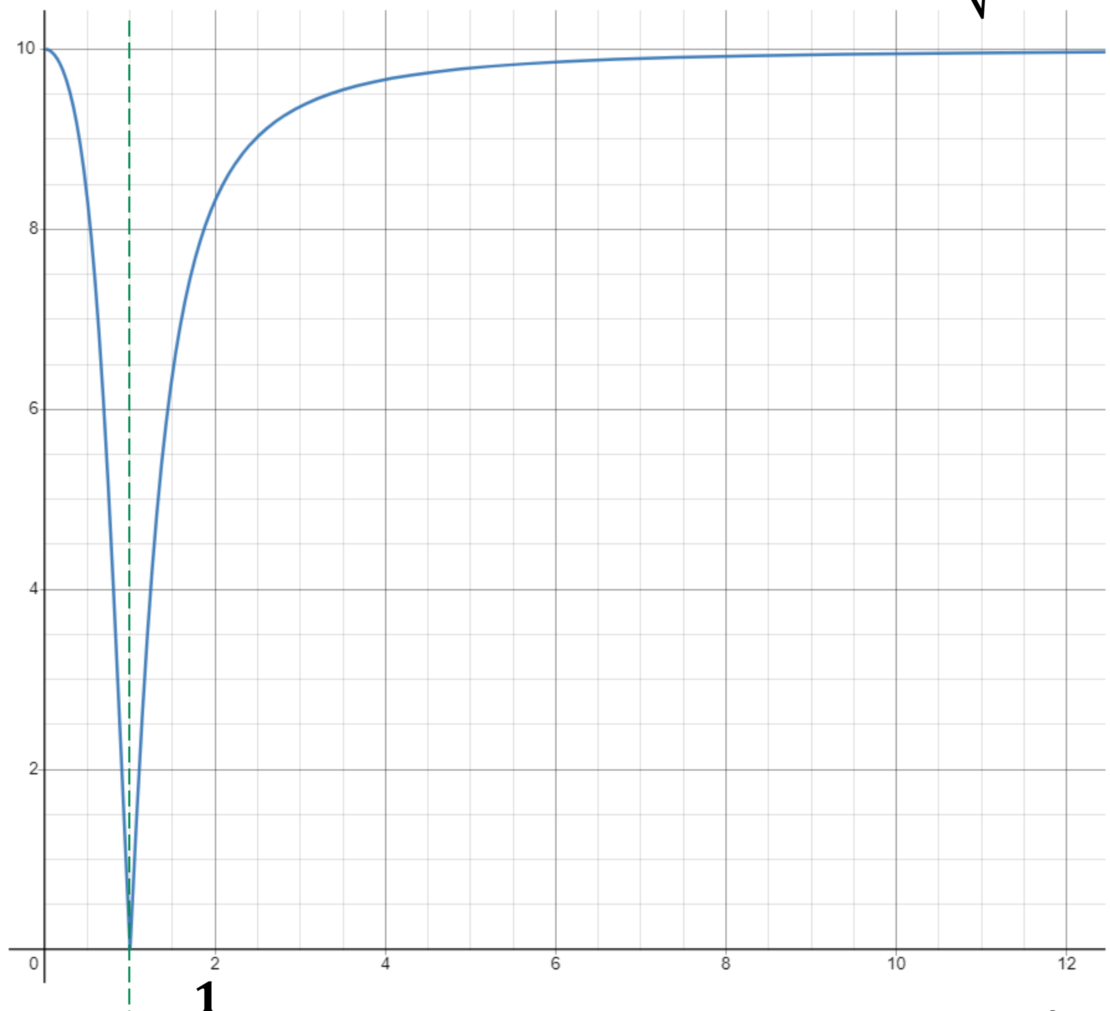
At the resonance condition, the reactance (parallel of L and C) behaves like an open circuit and no current can flow.

The peak value of the current is zero at resonance

$$|\tilde{I}| = \frac{|\tilde{V}|}{\sqrt{R^2 + \left(\frac{\omega L}{1-\omega^2 LC}\right)^2}}$$

$V_0 = 10\text{V}$ $R = 1\Omega$

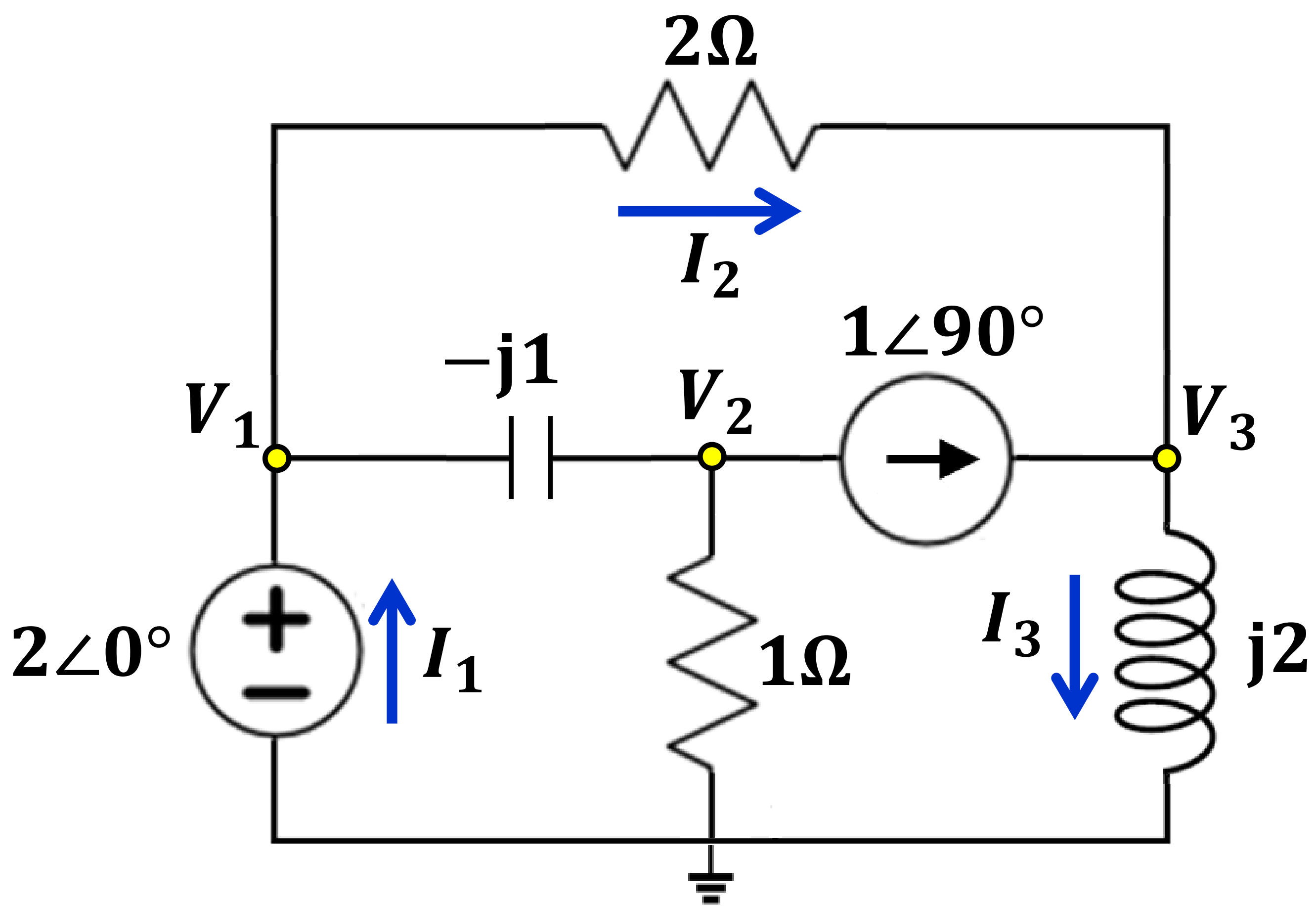
$L = 1\text{H}$ $C = 1\text{F}$



$$\omega_r = \frac{1}{\sqrt{LC}}$$

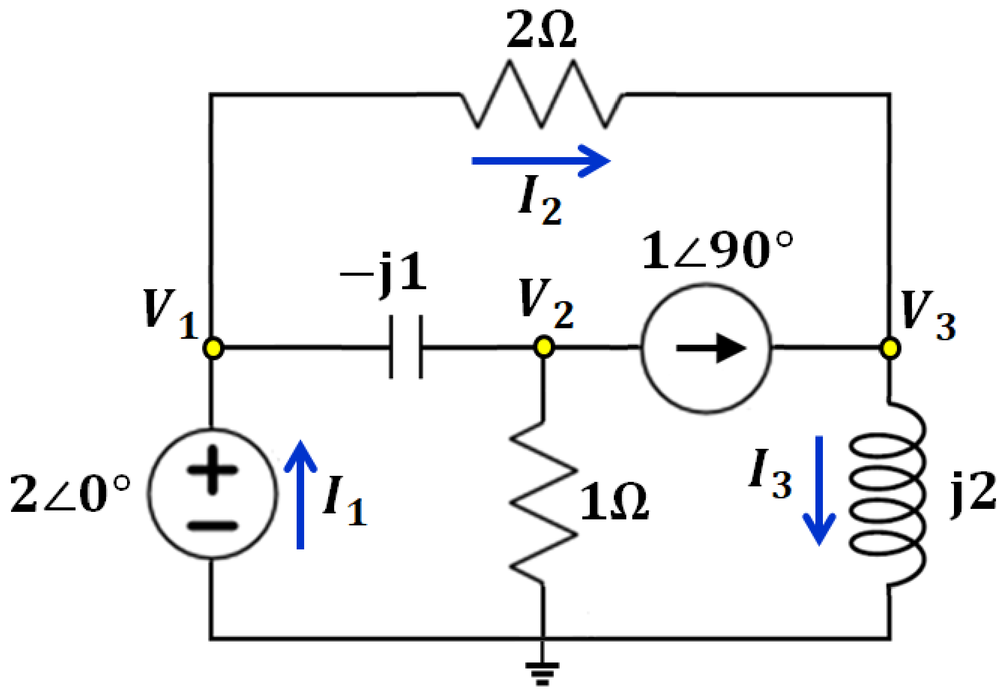
ω [rad/s]

Find phasors $V_1, V_2, V_3, I_1, I_2, I_3$
(We can drop the wavy hat ˜ now)



$$V_1 = 2\text{V}$$

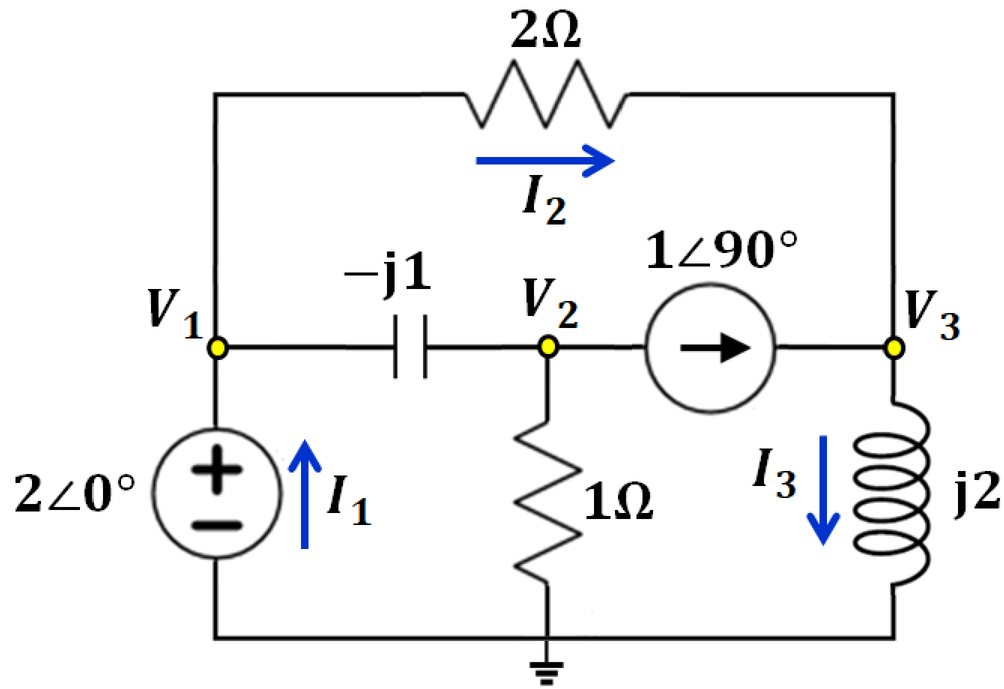
By inspection



Node 2 KCL

$$\frac{V_2 - V_1}{-\text{j}1} + \frac{V_2}{1} + 1\angle 90° = 0$$

$$V_1 = 2\text{V}$$



Node 2 KCL

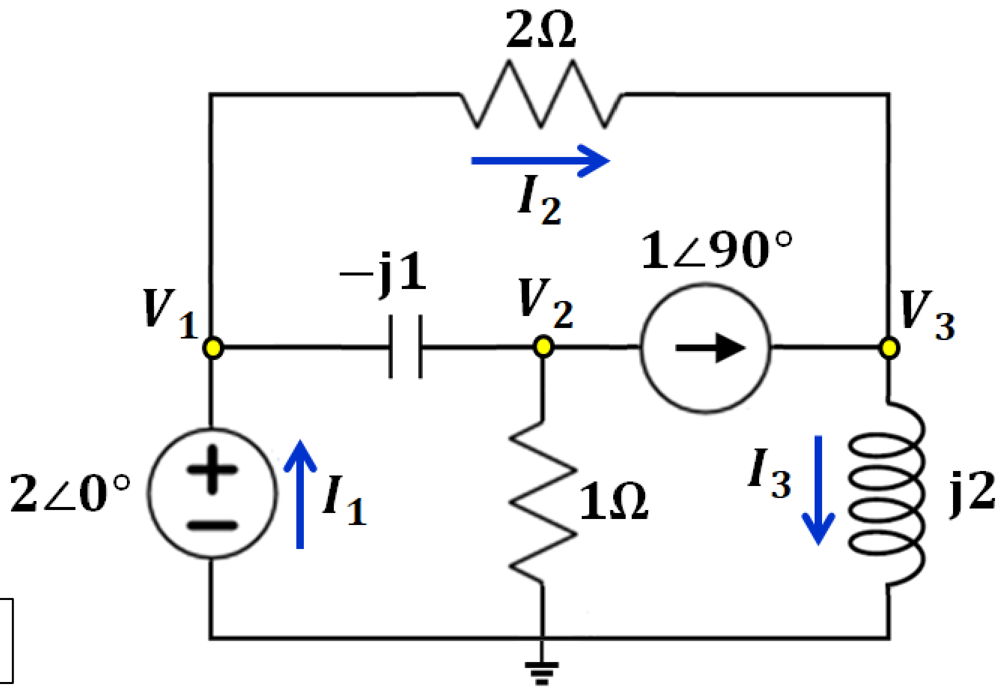
$$\frac{V_2 - V_1}{-\text{j}1} + \frac{V_2}{1} + 1\angle 90^\circ = 0$$

$$\text{j}V_2 - 2\text{j} + V_2 + \text{j}1 = 0 \quad \Rightarrow \quad V_2(1+\text{j}) = \text{j}1$$

$$V_2 = \frac{\text{j}1}{(1+\text{j})} = \frac{\text{j}1(1-\text{j})}{(1+\text{j})(1-\text{j})} = \frac{1+\text{j}}{2} = 0.5 + \text{j}0.5$$

$$V_1 = 2\text{V}$$

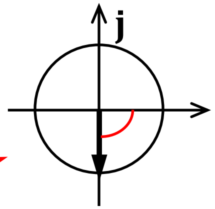
$$V_2 = 0.5 + j\,0.5\ \text{V}$$



V_1

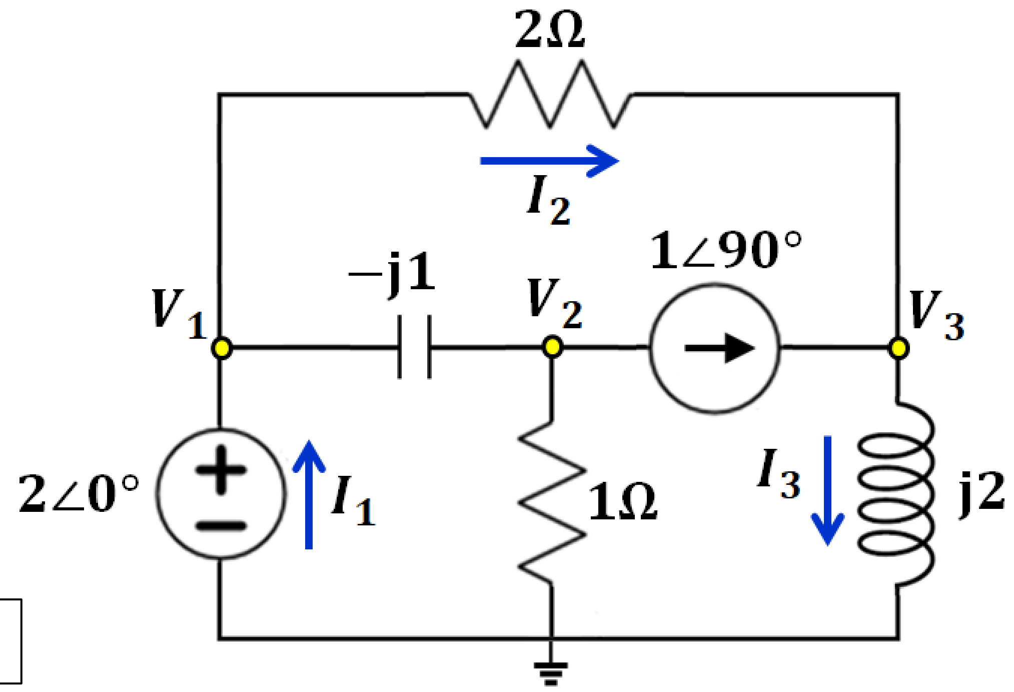
Node 3 KCL

$$\frac{V_3}{j2} + \frac{V_3 - 2}{2} - 1\angle 90° = 0$$



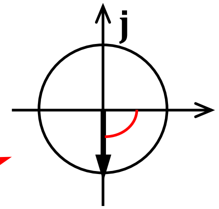
$$V_1 = 2\text{V}$$

$$V_2 = 0.5 + j\,0.5\ \text{V}$$



Node 3 KCL

$$\frac{V_3}{j2} + \frac{V_3 - \overset{V_1}{2}}{2} - 1\angle 90^\circ = 0$$



$$-j\frac{1}{2}V_3 + \frac{1}{2}V_3 - 1 - j1 = 0 \Rightarrow V_3 = \frac{1 + j1}{(1 - j1)/2}$$

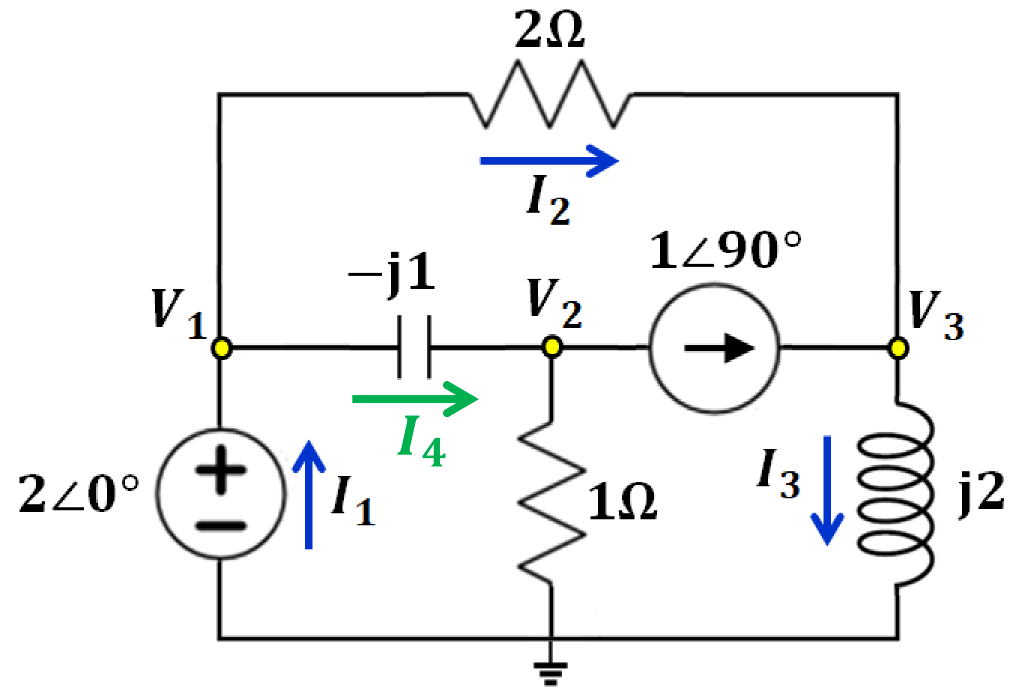
$$V_3 = \frac{\cancel{2}(1 + j1)(1 + j1)}{\cancel{2}} = j2\ \text{V}$$

$V_1 = 2\text{V}$

$V_2 = 0.5 + j\,0.5\ \text{V}$

$V_3 = j2\text{V}$

Currents



KCL node 1

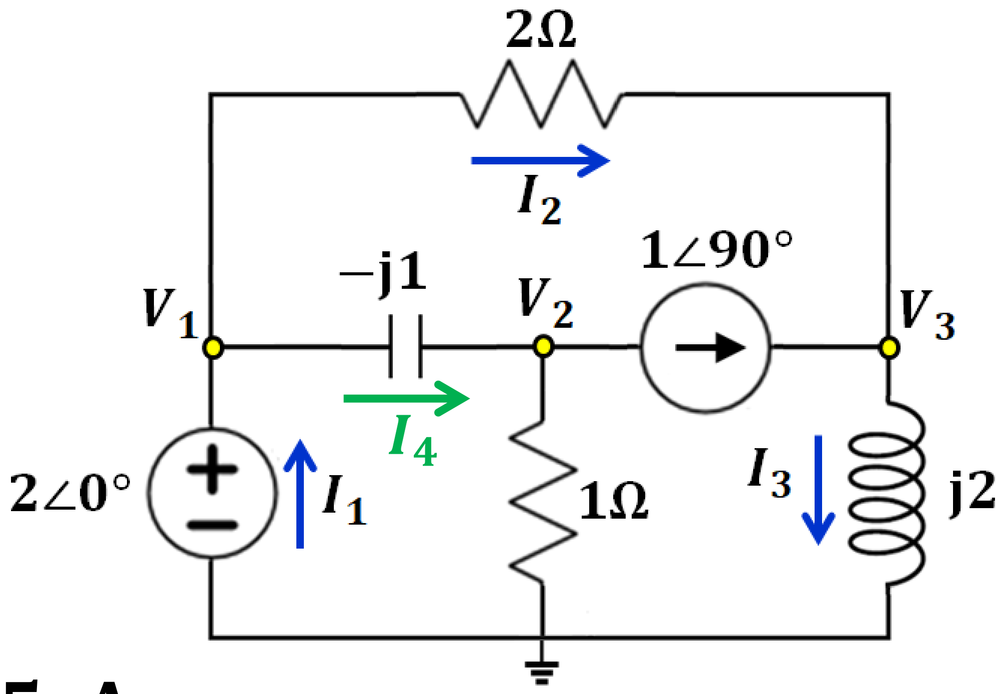
$$\underbrace{I_4 = I_1 - I_2}_{\text{KCL node 1}} = \frac{V_1 - V_2}{-j1} = \frac{2 - (0.5 + j0.5)}{-j1}$$

$$= 0.5 + j1.5\ \text{A}$$

$$V_1 = 2\text{V}$$

$$V_2 = 0.5 + j\,0.5\ \text{V}$$

$$V_3 = j2\text{V}$$



Currents

$$I_4 = I_1 - I_2 = 0.5 + j1.5\ \text{A}$$

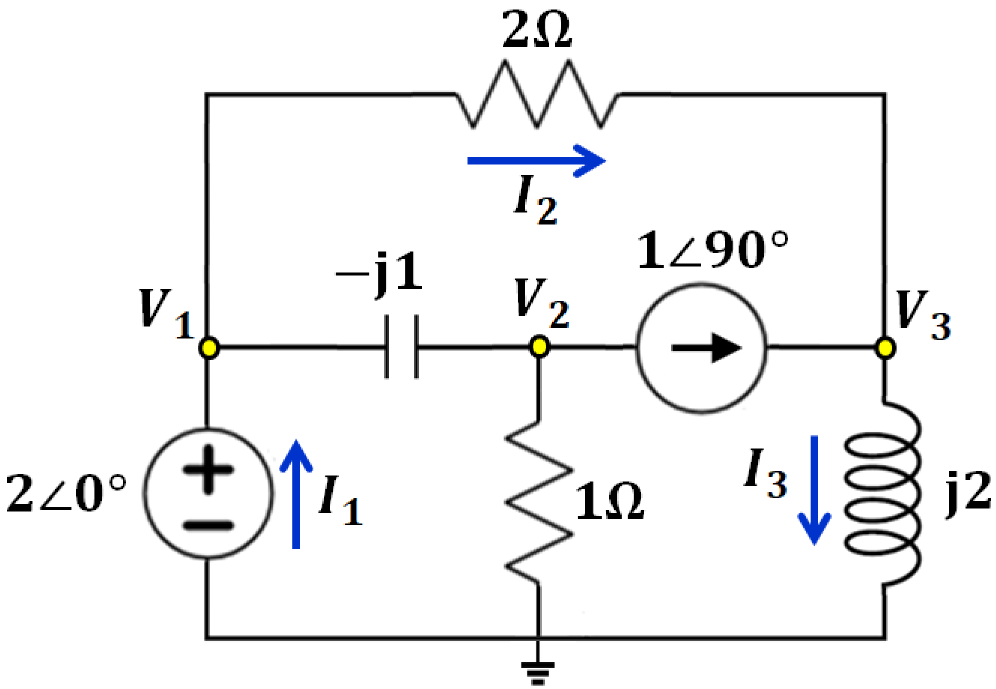
$$I_2 = \frac{V_1 - V_3}{2} = \frac{2 - j2}{2} = 1 - j\ \text{A}$$

$$I_1 = I_4 + I_2 = (0.5 + j1.5) + (1 - j)$$
$$= 1.5 + j0.5\ \text{A}$$

$$V_1 = 2\text{V}$$

$$V_2 = 0.5 + \text{j}\,0.5\ \text{V}$$

$$V_3 = \text{j}2\text{V}$$



Currents

$$I_1 = 1.5 + \text{j}0.5\ \text{A}$$

$$I_2 = 1 - \text{j}\ \ \text{A}$$

KCL at Node 3

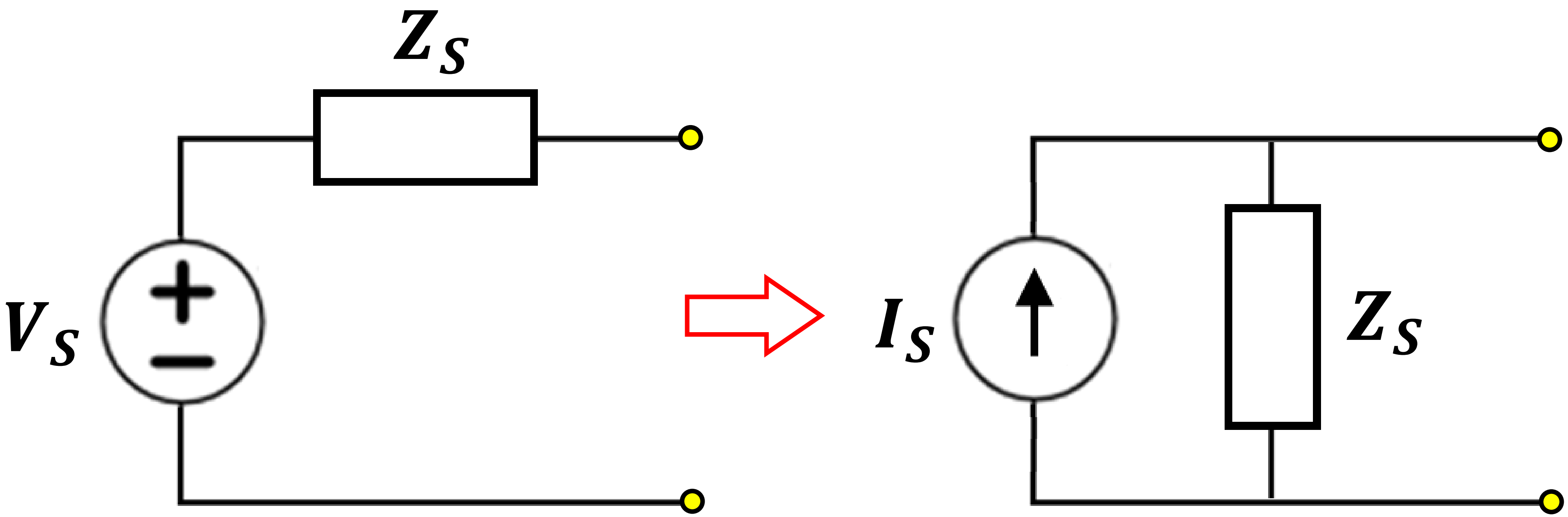
$$I_3 - I_2 - 1\angle 90° = 0$$

$$I_3 - (1 - \text{j}1) - \text{j}1 = 0$$

Also: $I_3 = V_3/\text{j}2 = \text{j}2/\text{j}2 = 1$ → $I_3 = 1\ \text{A}$

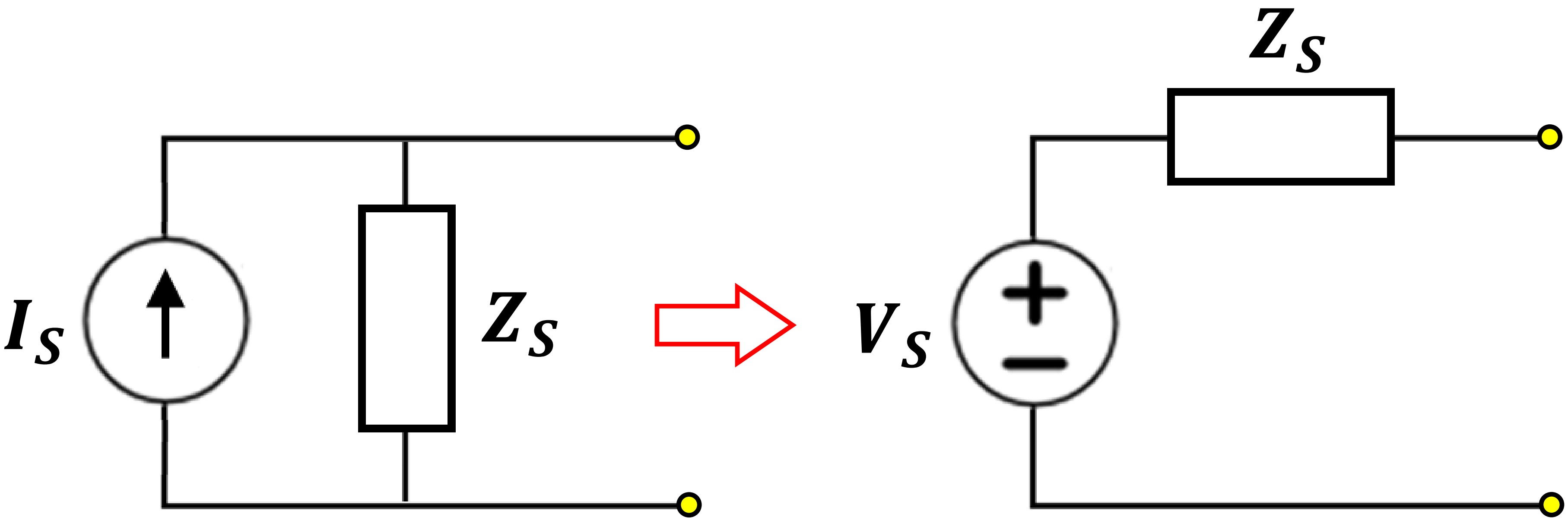
Source Transformations

The approach we used before works for phasors, too.



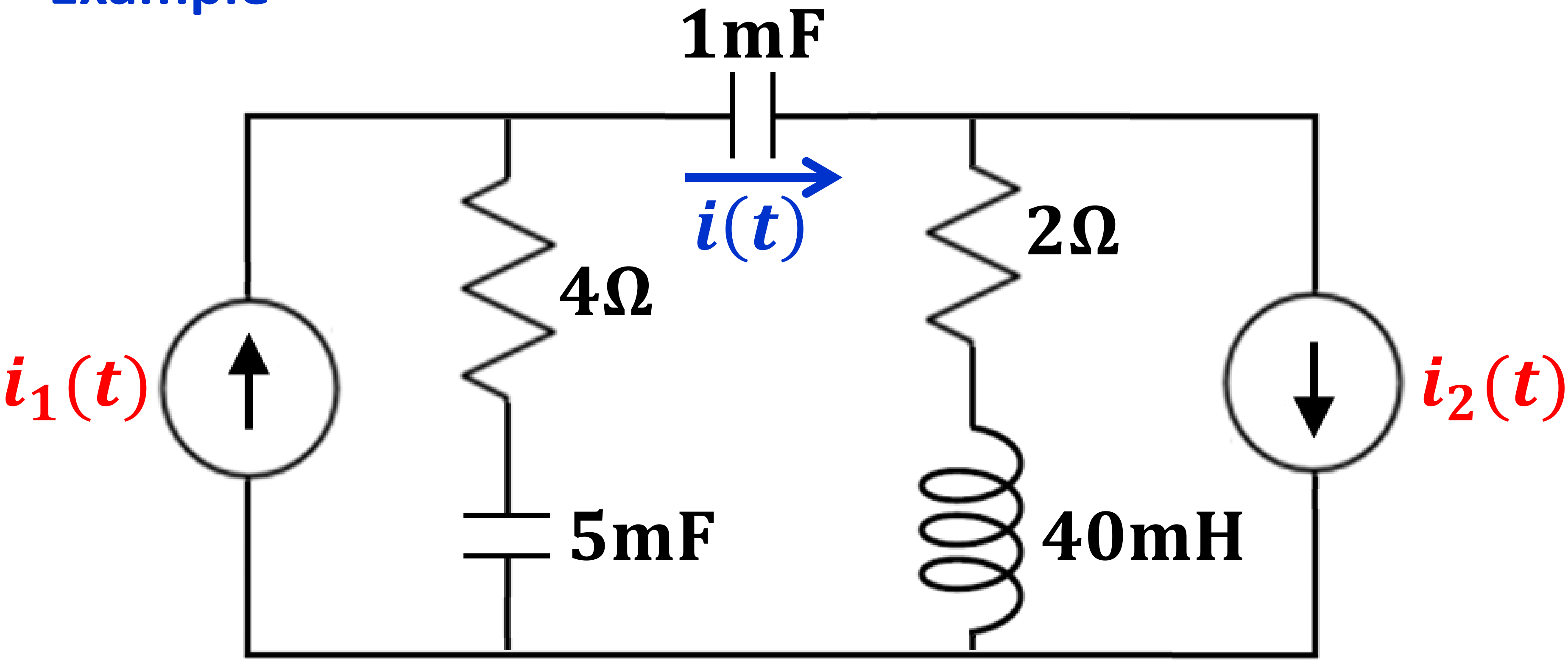
$$I_S = \frac{V_S}{Z_s}$$

Source Transformations



$$V_S = I_S Z_s$$

Example



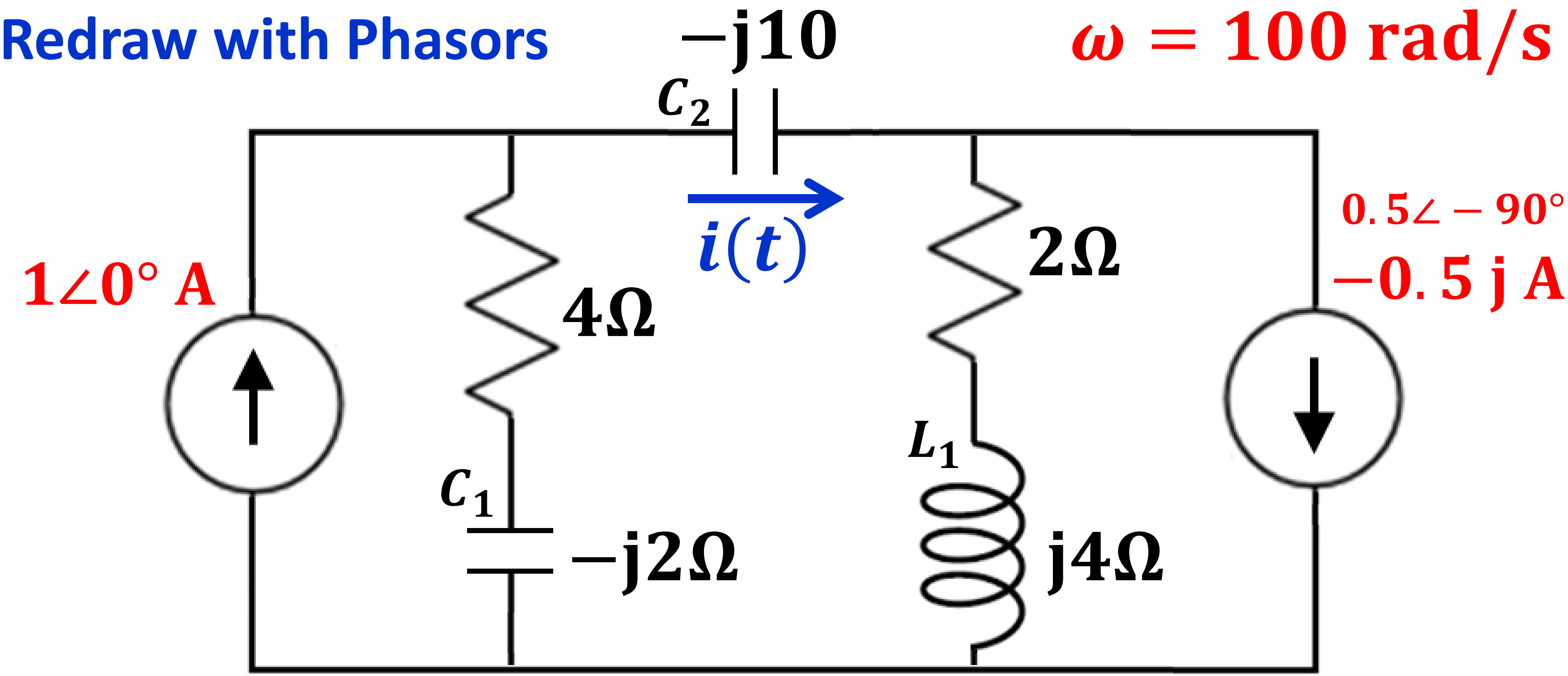
$$i_1(t) = 1\cos(100t)\ \text{A}$$

$$i_2(t) = 0.5\cos(100t - 90°)\ \text{A}$$

Find $i(t)$

Redraw with Phasors

$\omega = 100 \text{ rad/s}$

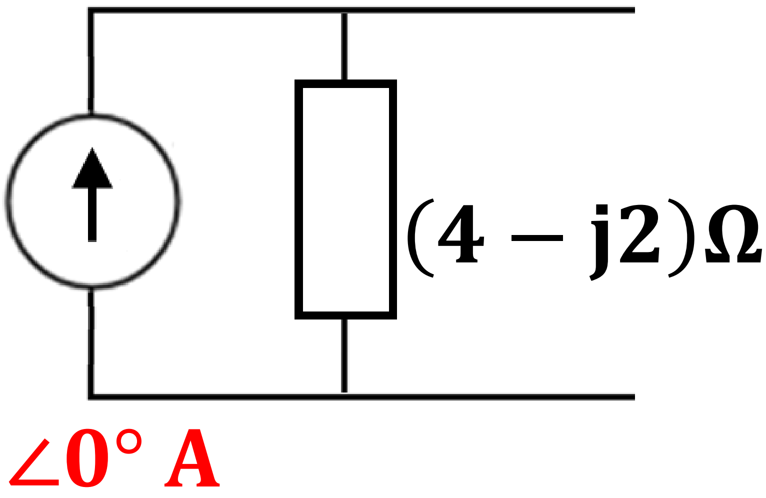


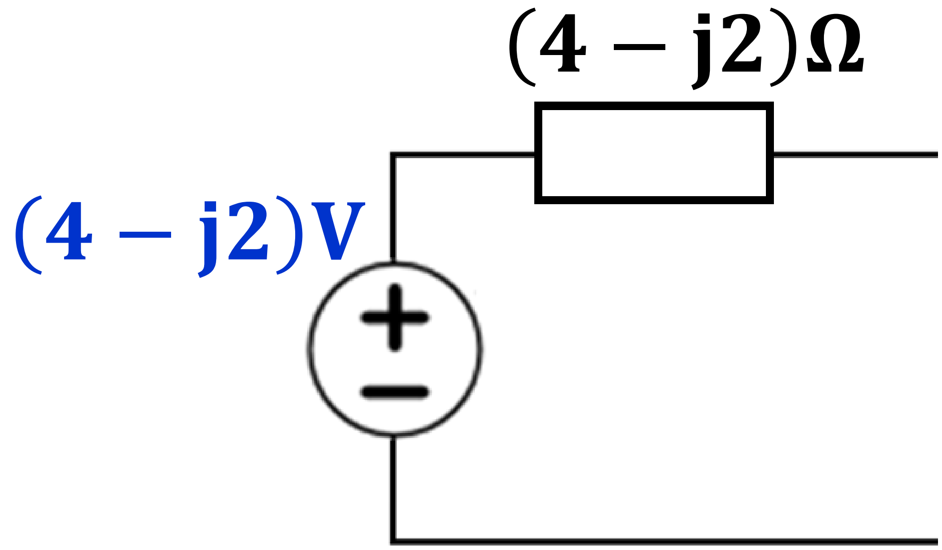
$$C_1 = 5\text{mF} \rightarrow \quad -j\frac{1}{\omega C_1} = -j\frac{1}{100 \times 5 \times 10^{-3}} = -j2\Omega$$

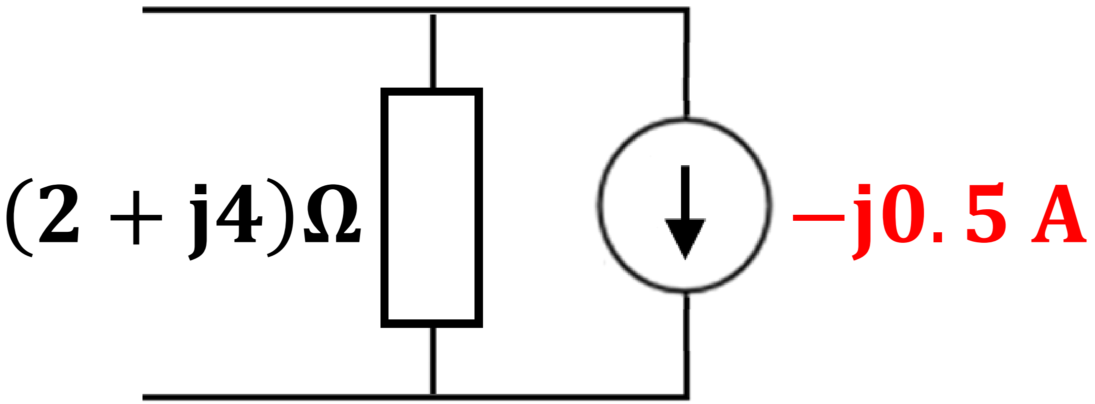
$$C_2 = 1\text{mF} \rightarrow \quad -j\frac{1}{\omega C_2} = -j\frac{1}{100 \times 10^{-3}} = -j10\Omega$$

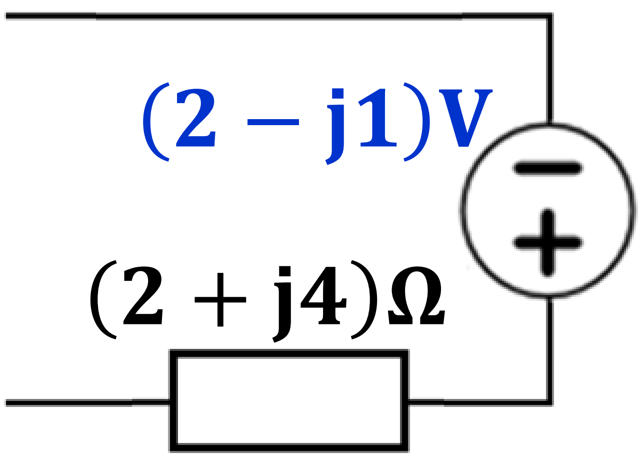
$$L_1 = 40\text{mH} \rightarrow \quad j\omega L_1 = j100 \times 40 \times 10^{-3} = j4\ \Omega$$

Source Transformations

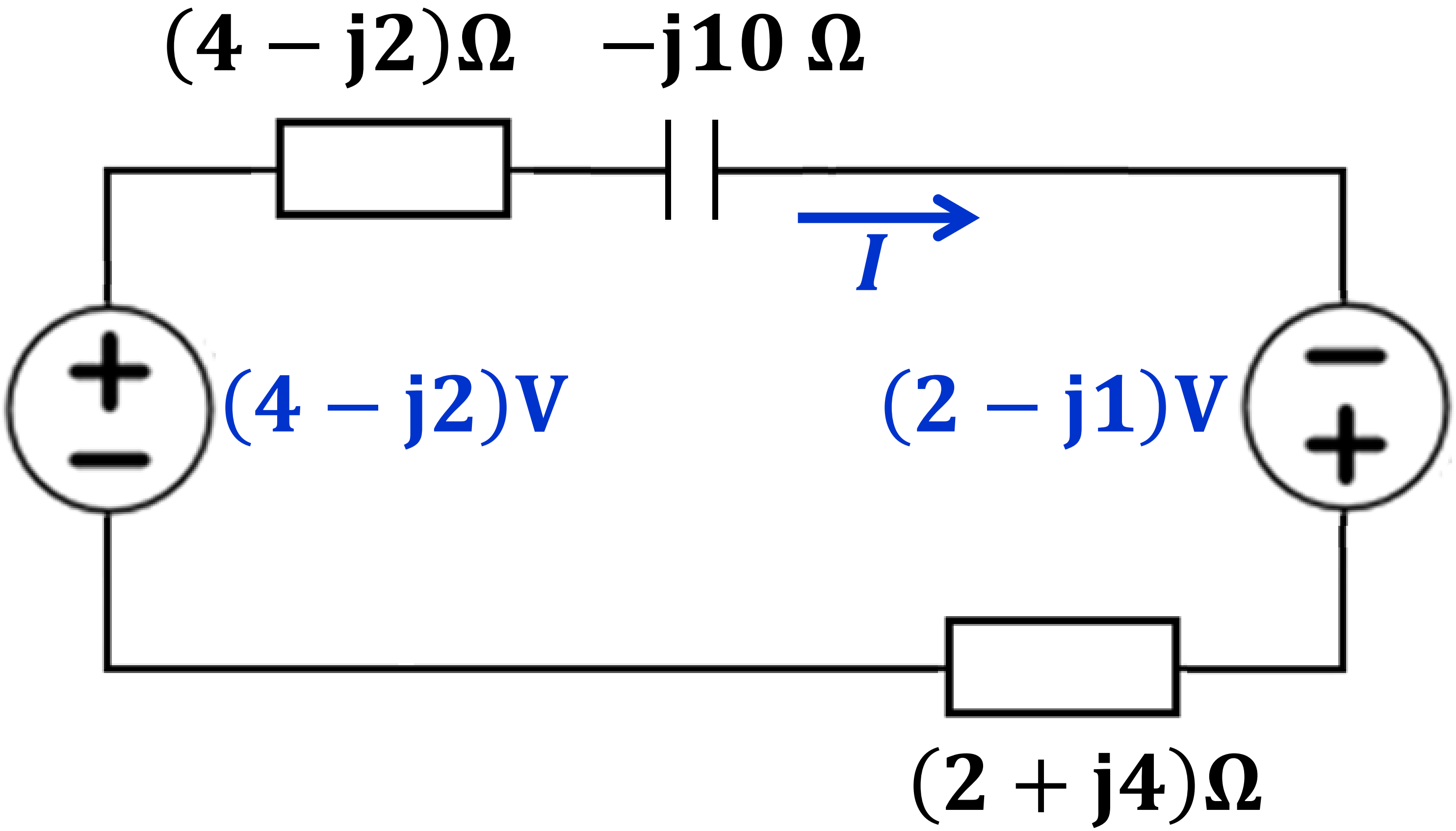








$$V_T = -j0.5(2 + j4) = (2 - j1)\text{ V}$$

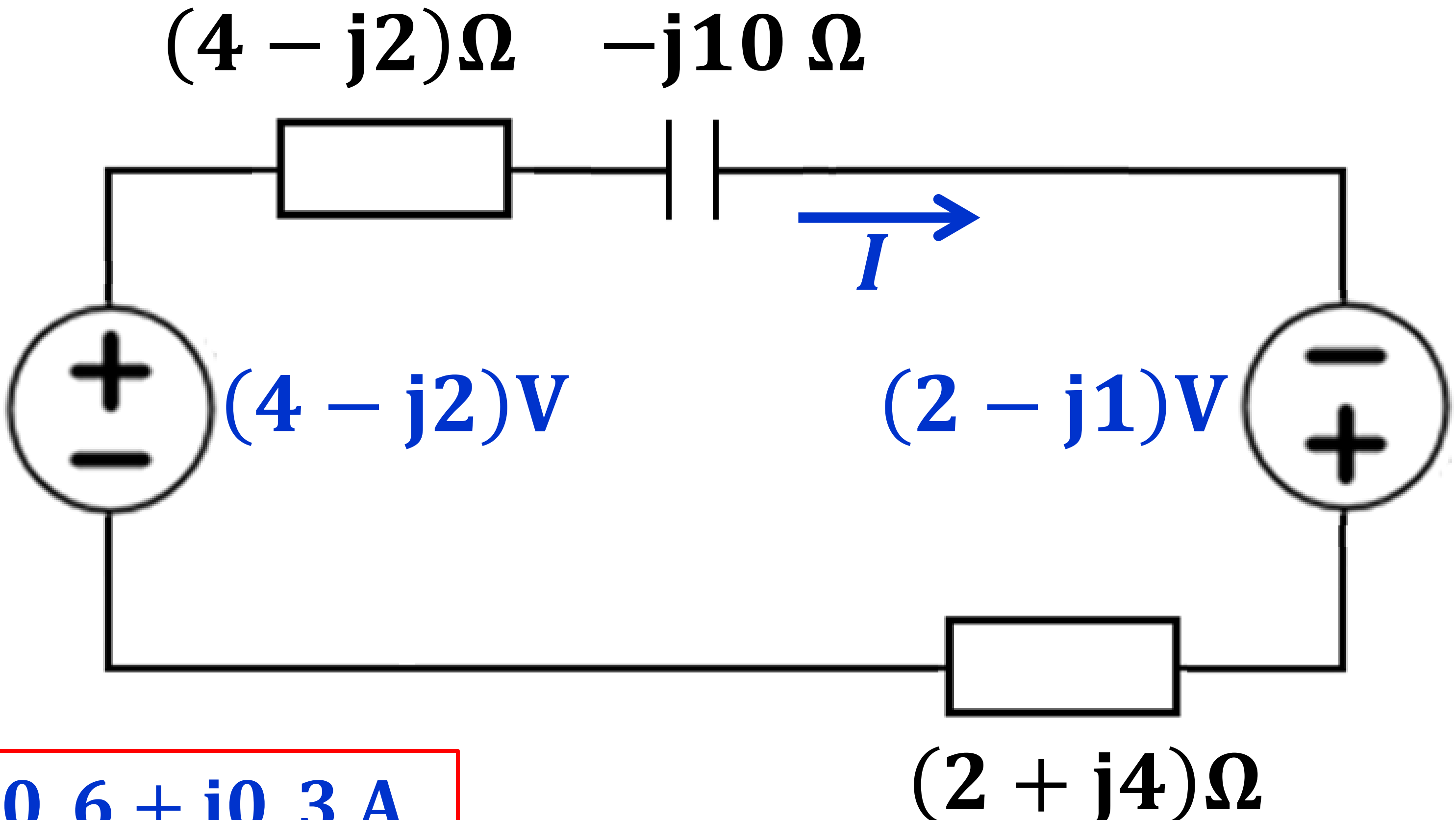


KVL

$$-(4-j2)-(2-j1)+I[(4-j2)-j10+(2+j4)]=0$$

$$\Rightarrow (6-j3)=I(6-j8)$$

$$I=\frac{6-j3}{6-j8}=\frac{(6-j3)(6+j8)}{100}=\frac{60+j30}{100}=0.6+j0.3\text{ A}$$



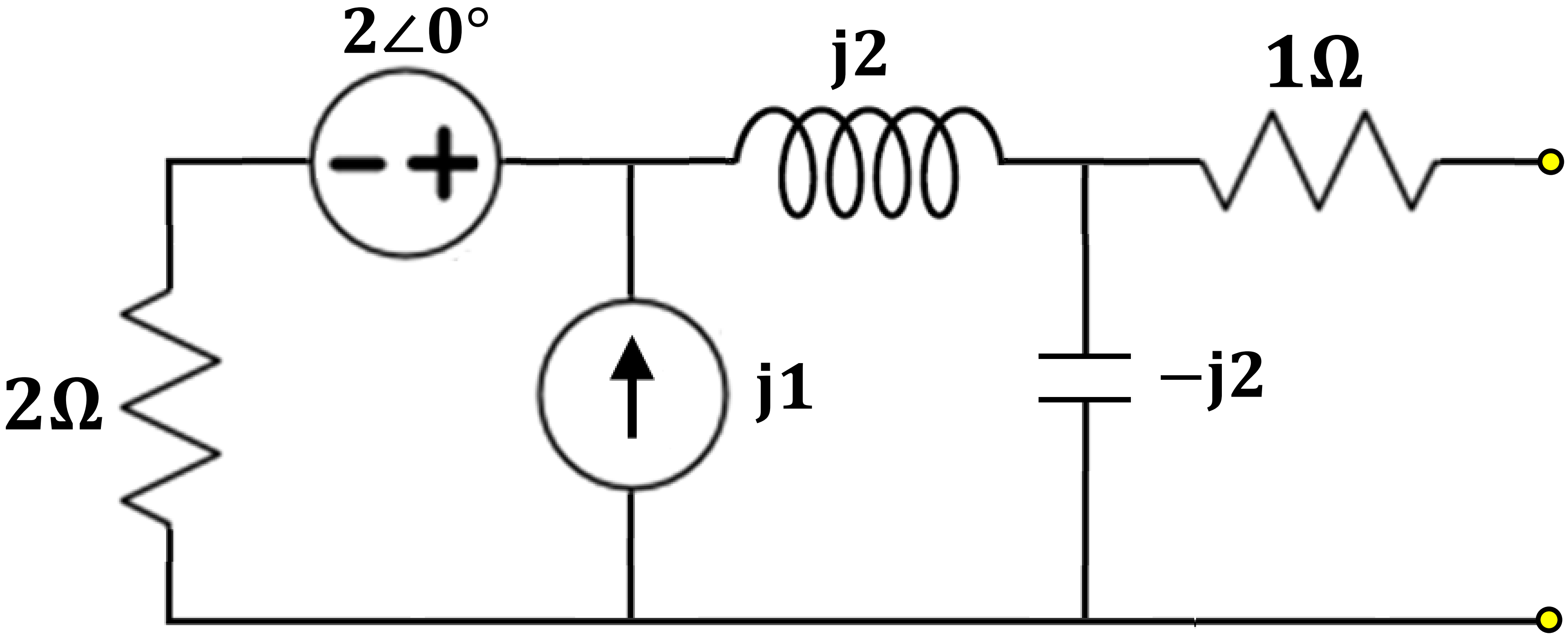
$$I = 0.6 + j0.3\ \mathrm{A}$$

$$I = \sqrt{0.6^2 + 0.3^2}\exp\left(j\tan^{-1}\left(\frac{0.3}{0.6}\right)\right)$$

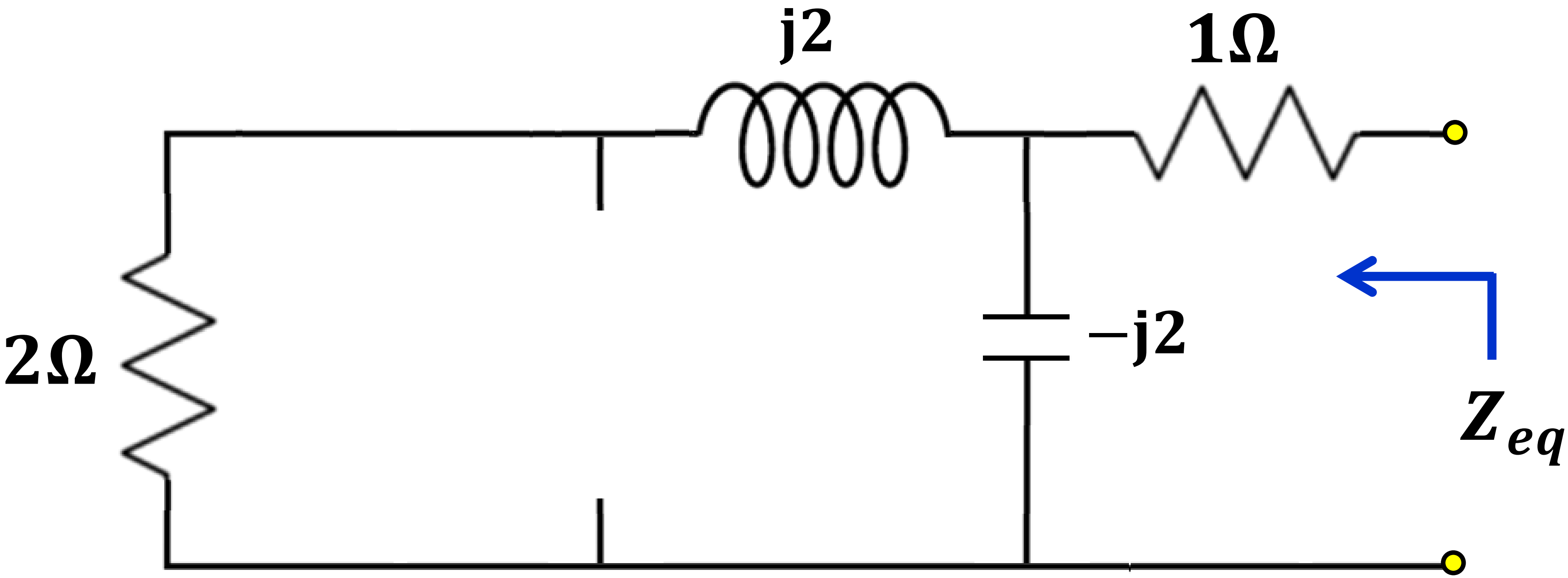
$$= 0.6708\angle 0.46365\mathrm{rad} = 0.6708\angle 26.57°$$

$$\Rightarrow\quad i(t) = 0.6708\cos(100t + 26.57°)$$

Find the Thevenin equivalent circuit



Equivalent impedance

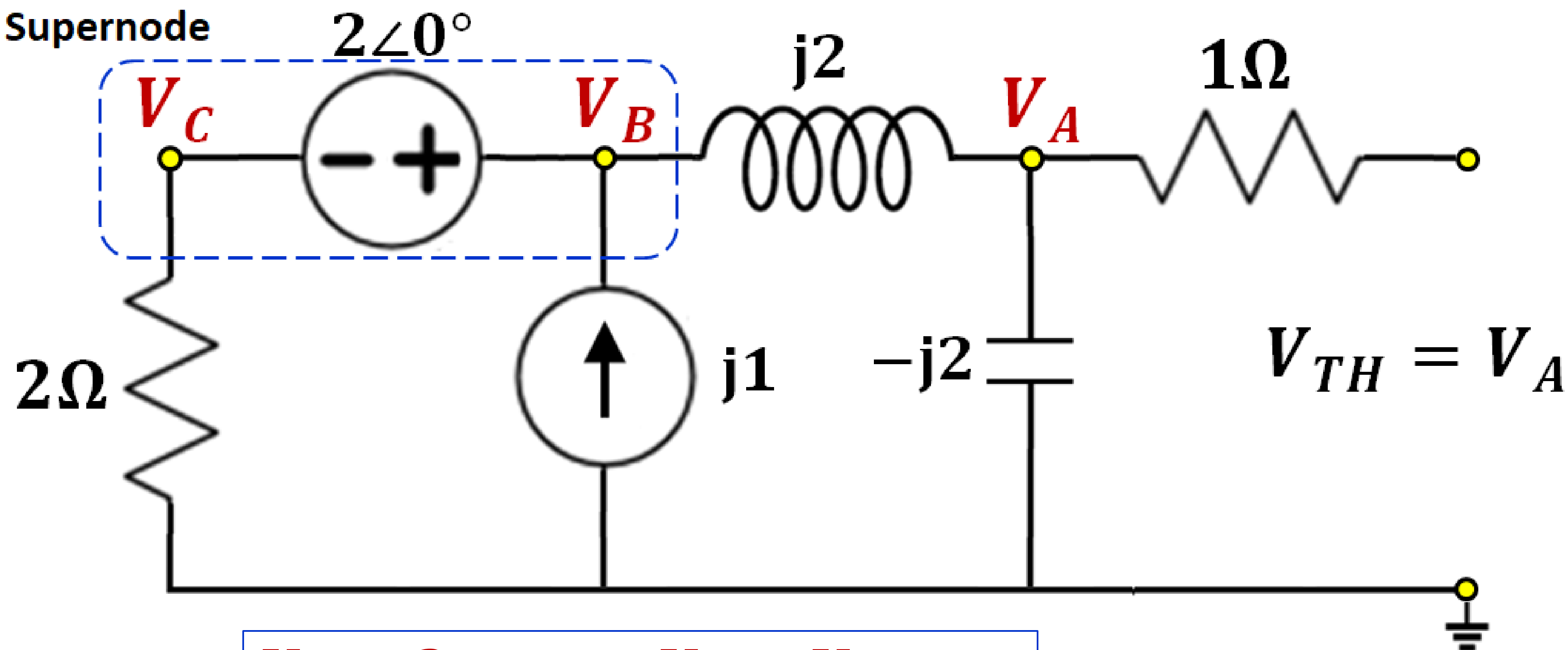


$$Z_{eq} = 1\Omega + (2\Omega + j2\Omega)//(-j2\Omega)$$

$$Z_{eq} = 1 + \left[\frac{1}{(2 + j2)} + \frac{1}{(-j2)}\right]^{-1} = 1 + \frac{(2 + j2)(-j2)}{2 + j2 - j2}$$

$$Z_{eq} = 1 + \frac{4 - j4}{2} = 3 - j2\ \Omega$$

Find open circuit voltage V_{TH}



(Supernode) $$\frac{V_B - 2}{2\Omega} - j1 + \frac{V_B - V_A}{j2\Omega} = 0$$

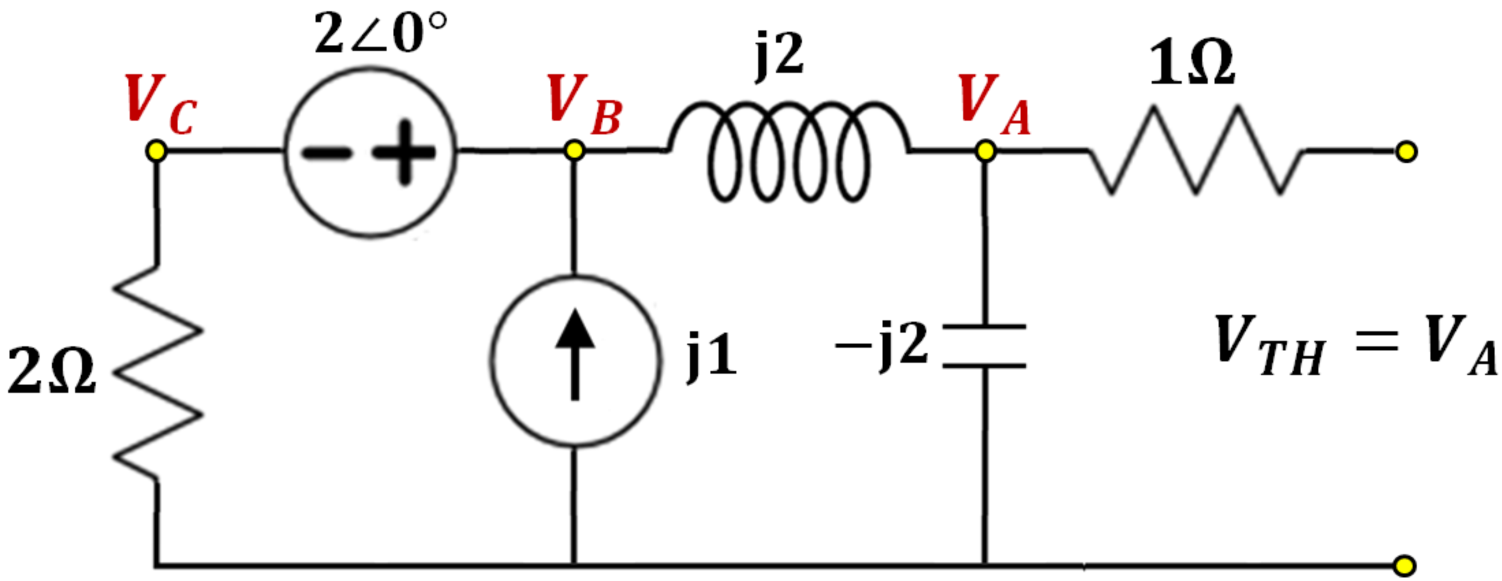
$$V_C = V_B - 2\text{V}$$

(Node A) $$\frac{V_A - V_B}{j2\Omega} + \frac{V_A}{-j2\Omega} = 0$$

⇒ $$V_B = 0$$

Substituting $V_B = 0$ in Supernode KCL: $$V_A = 2 - j2\ \text{V}$$

Thevenin equivalent circuit



$$Z_{TH} = 3 - j2\ \Omega$$

$$V_{TH} = 2 - j2\ \text{V}$$