ECE 205 “Electrical and Electronics Circuits”

Spring 2024 – LECTURE 34

MWF – 12:00pm

Prof. Umberto Ravaioli

2062 ECE Building

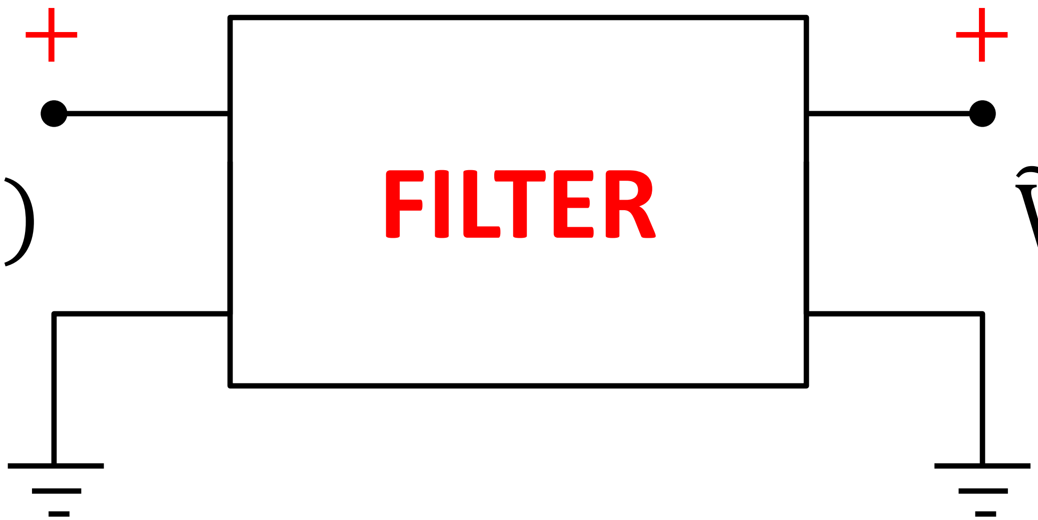
Lecture 34 – Summary

Learning Objectives

1. Frequency Response of Circuits
2. Low-Pass Passives filters
3. High-Pass Passive filters

Filter

A circuit which manipulates a signal, typically by changing the relative amplitudes of the frequency components.



$$\tilde{V}_{in}(\omega) \qquad \tilde{V}_{out}(\omega)$$

We will consider filters (systems) which are “single-input” and “single-output,” consisting of “linear” and “time-invariant” circuits.

Fourier transform

Converts a time domain signal $V(t)$ to a frequency domain signal $\tilde{V}(\omega)$

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} V(t)\, e^{-j\omega t}\, dt$$

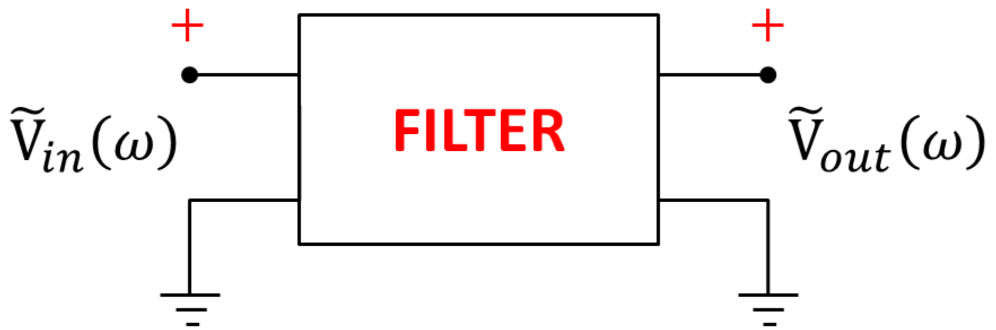
In general, the Fourier Transform is a complex function

Anti-Transform

$$V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(\omega)\, e^{j\omega t}\, d\omega$$

Transfer Function

The relationship linking the frequency-dependent input and output



$$\tilde{V}_{out}(\omega) = H(\omega)\,\tilde{V}_{in}(\omega)$$

$$H(\omega) = \frac{\tilde{V}_{out}(\omega)}{\tilde{V}_{in}(\omega)} = |H(\omega)|\,\angle\theta(\omega)$$

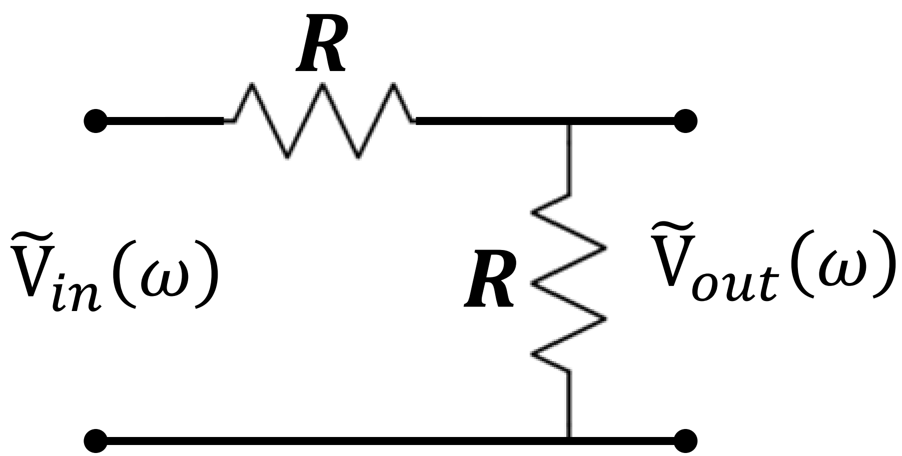
Transfer Function

Magnitude Response

Phase Response

1 – Voltage Divider

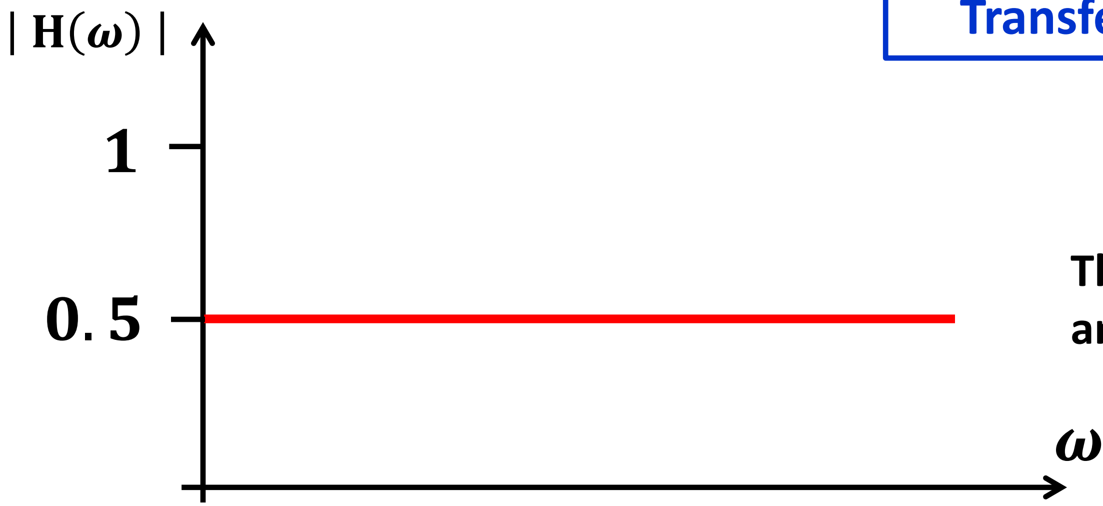
The voltage divider is a very simple filter



$$\tilde{V}_{out}(\omega) = \frac{1}{2}\tilde{V}_{in}(\omega)$$

$$H(\omega) = |H(\omega)| = \frac{1}{2}$$

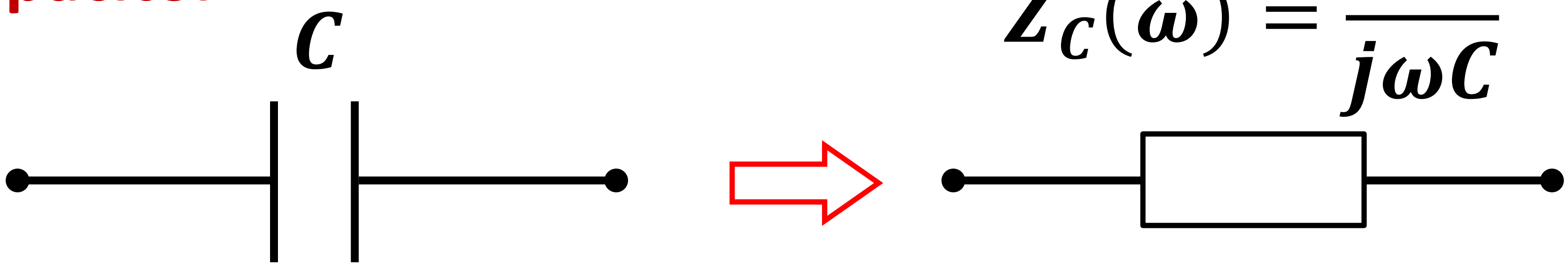
Transfer Function



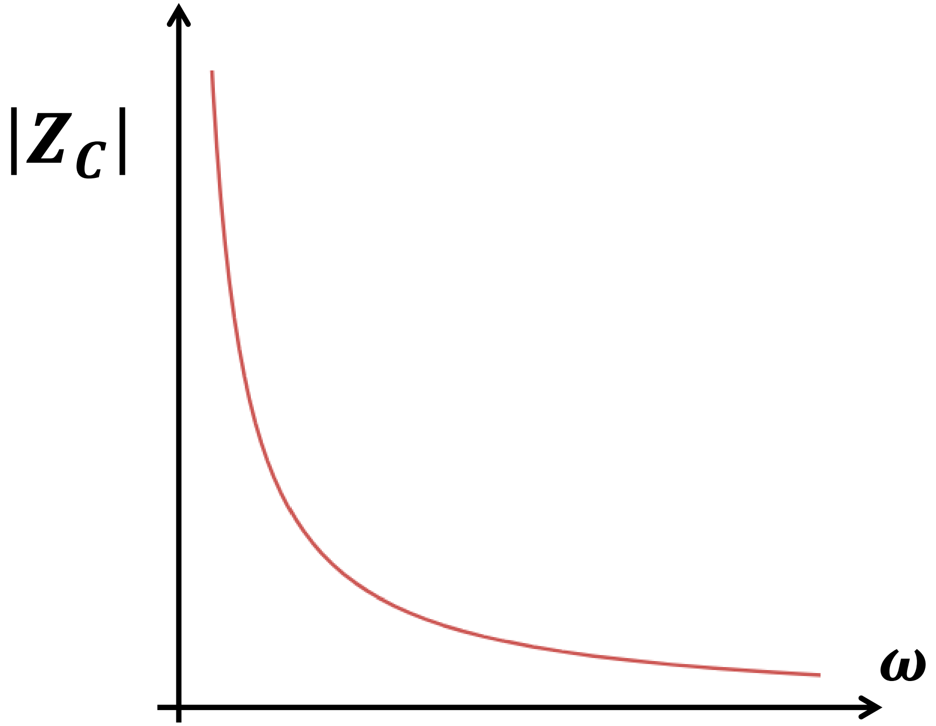
This circuit works as an "attenuator"

Behavior of Reactive Circuit Elements

Capacitor

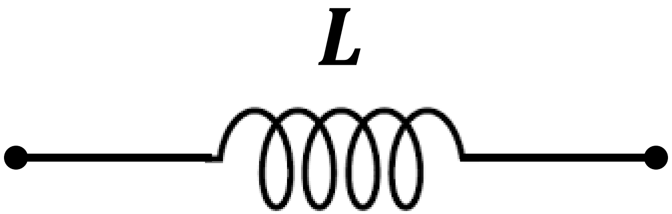


$$Z_C(\omega) = \frac{1}{j\omega C}$$

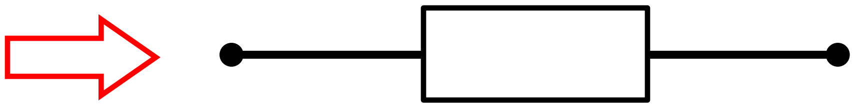


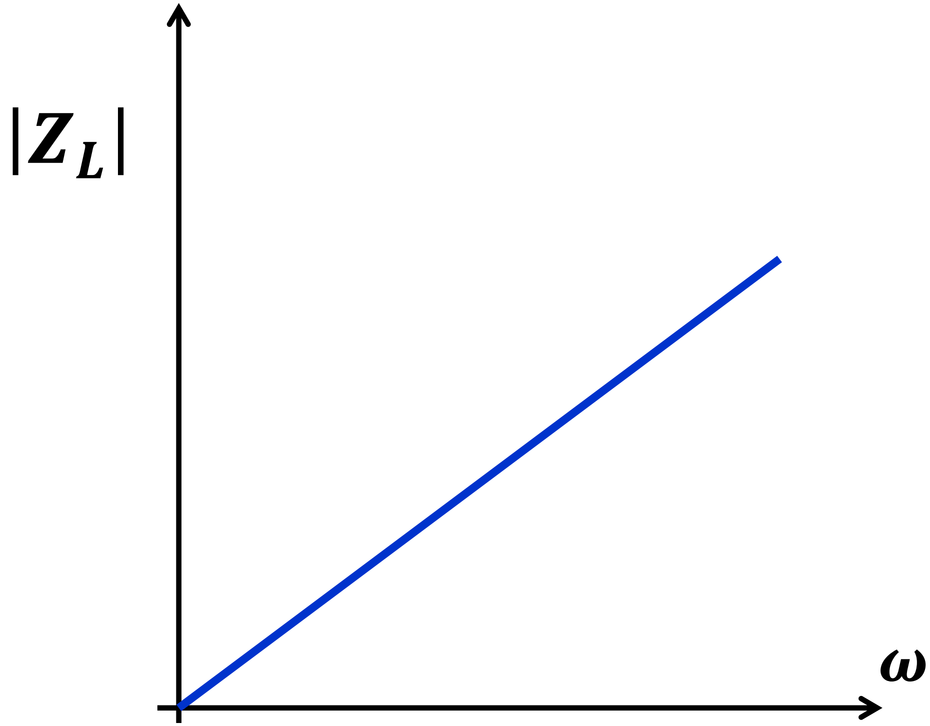
Behavior of Reactive Circuit Elements

Inductor



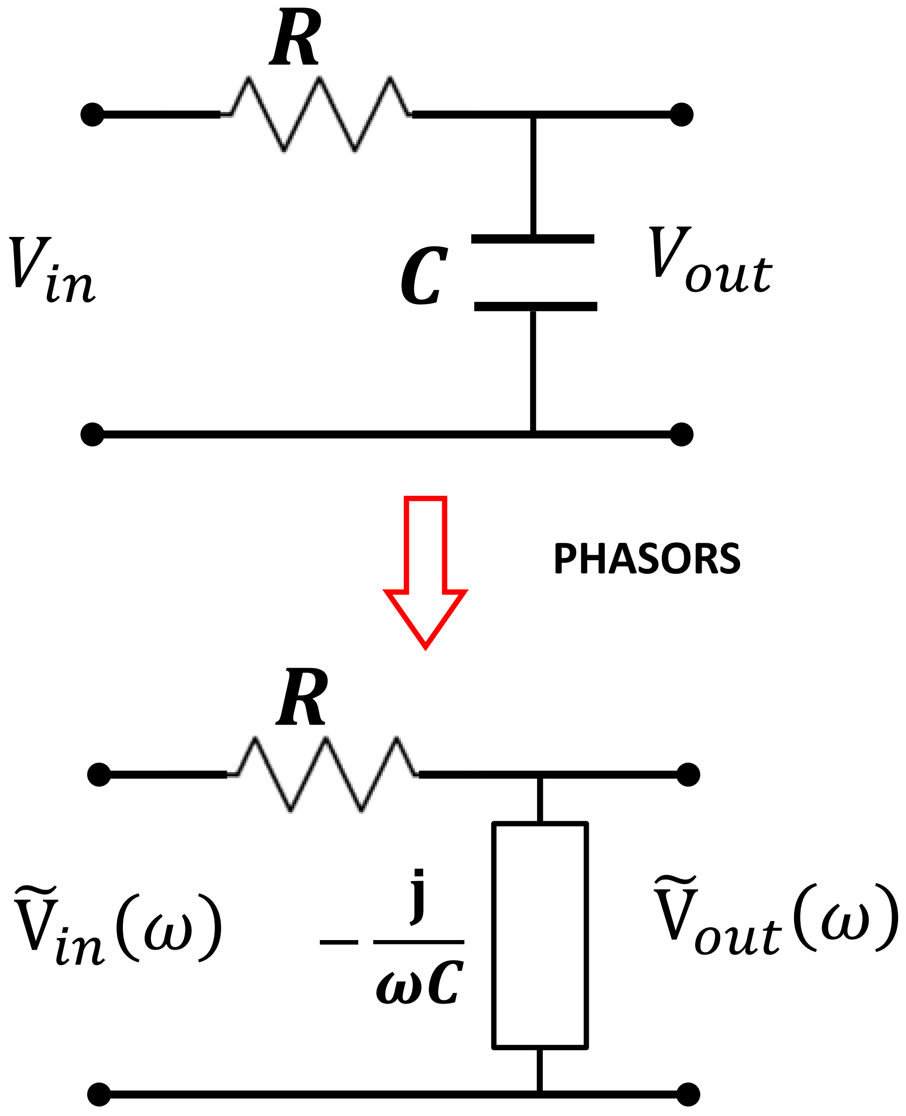
$$Z_L(\omega) = \mathrm{j}\omega L$$

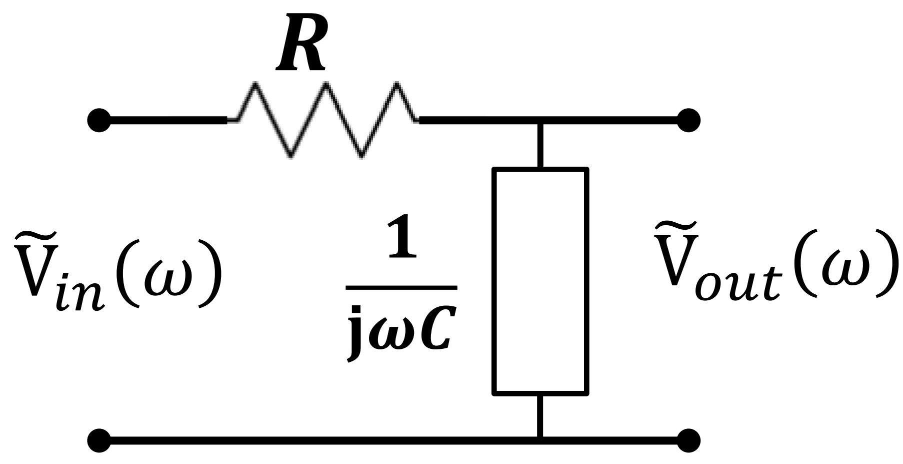




2 – Low Pass RC filter

RC filter
(1st order)





Let the input be a phasor of the form

$$\tilde{V}_{in}(\omega) = V_I \angle 0°$$

$$\tilde{V}_{out}(\omega) = V_I \angle 0° \frac{1/j\omega C}{R + 1/j\omega C} = \overbrace{V_I \angle 0°}^{\tilde{V}_{in}(\omega)} \frac{1}{1 + j\omega RC}$$

$$\frac{\tilde{V}_{out}(\omega)}{\tilde{V}_{in}(\omega)} = H(\omega) = \frac{1}{1 + j\omega RC}$$

Transfer Function

$$\mathbf{H}(\omega) = \frac{1}{1 + j\omega RC}$$

$$\mathbf{H}(\omega) = \frac{1 - j\omega RC}{(1 + j\omega RC)(1 - j\omega RC)} = \underbrace{\frac{1 - j\omega RC}{1 + (\omega RC)^2}}_{\text{Cartesian Form}}$$

Magnitude

$$|\mathbf{H}(\omega)| = \frac{1}{|1 + j\omega RC|}$$

$$|\mathbf{H}(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

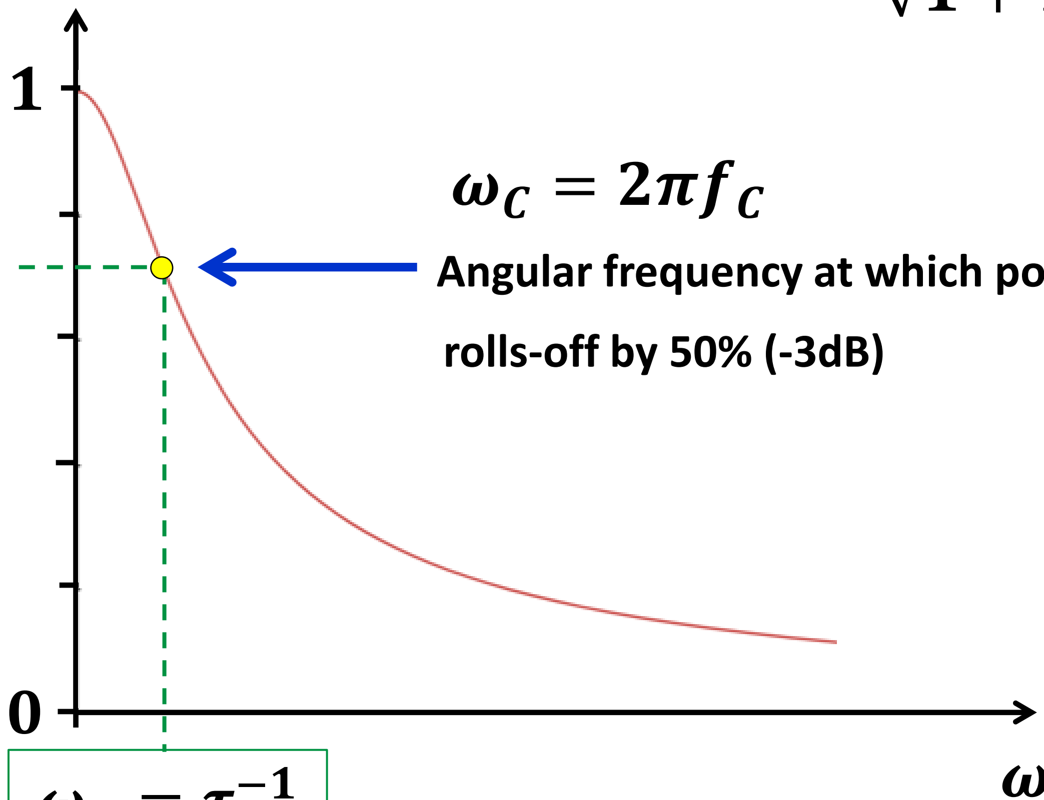
Magnitude of $H(\omega)$ for RC low-pass filter

At $\omega = \omega_C$: $R = 1/\omega C$

$$|\mathrm{H}(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$\omega_C RC = 1$ → $|\mathrm{H}(\omega_C)| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$

$|\mathrm{H}(\omega)|$



$\frac{1}{\sqrt{2}} \approx 0.707$

$\omega_C = 2\pi f_C$

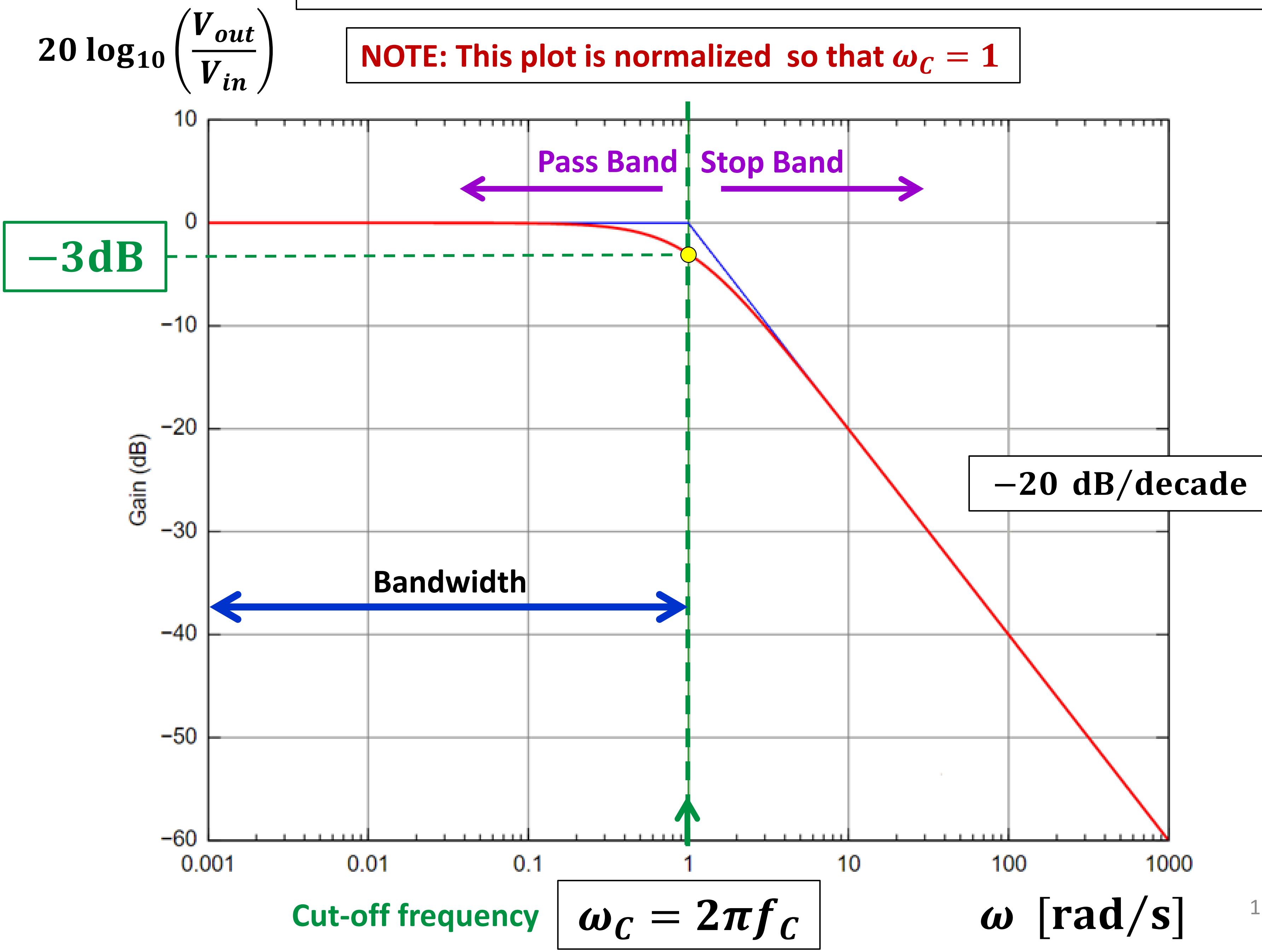
Angular frequency at which power $P \propto |H(\omega)|^2$ rolls-off by 50% (-3dB)

$\tau = RC$

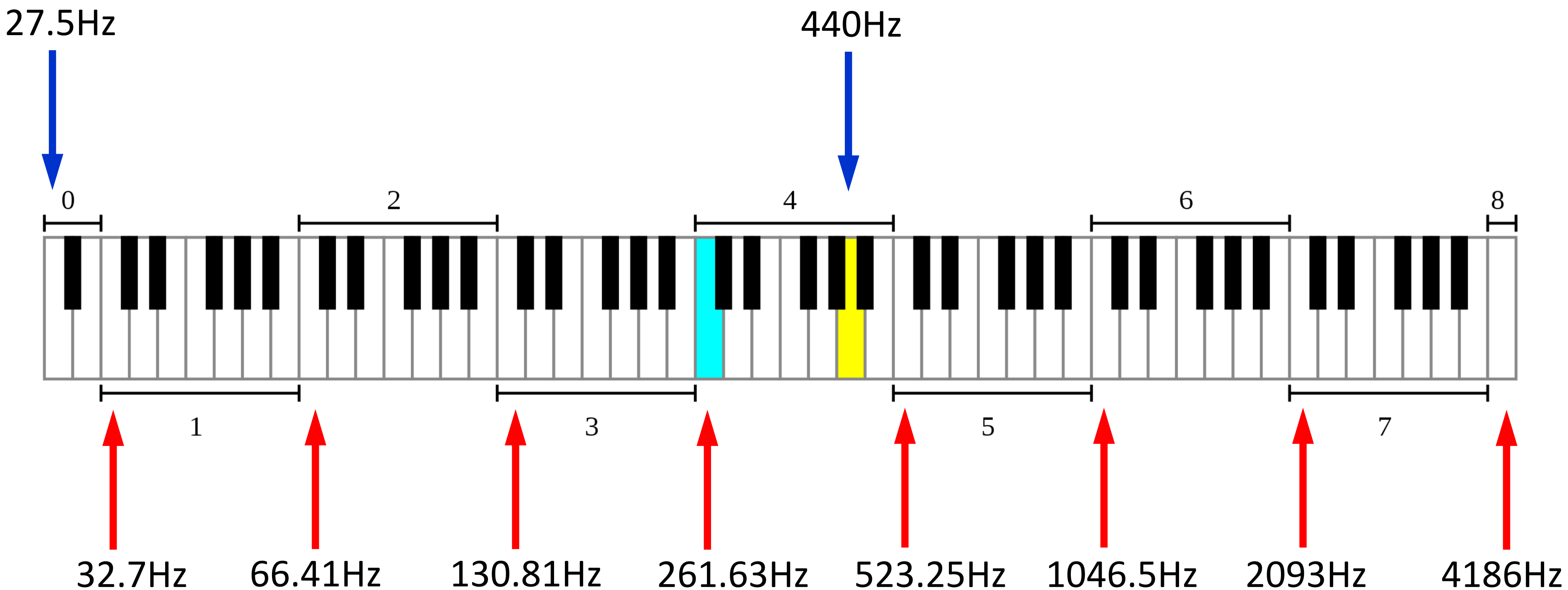
$\omega_C = \tau^{-1}$

log-decibel representation – Bode Plot for magnitude

NOTE: This plot is normalized so that $\omega_C = 1$



The piano keyboard uses octaves (instead of decades)



Phase of $H(\omega)$ for RC low-pass filter

$$\mathrm{H}(\omega) = \frac{1 - \mathrm{j}\omega RC}{(1 + \mathrm{j}\omega RC)(1 - \mathrm{j}\omega RC)} = \underbrace{\frac{1 - \mathrm{j}\omega RC}{1 + (\omega RC)^2}}_{\text{Cartesian Form}}$$

Phase of $H(\omega)$ for RC low-pass filter

$$\mathrm{H}(\omega) = \frac{1 - \mathrm{j}\omega RC}{(1 + \mathrm{j}\omega RC)(1 - \mathrm{j}\omega RC)} = \underbrace{\frac{1 - \mathrm{j}\omega RC}{1 + (\omega RC)^2}}_{\text{Cartesian Form}}$$

$$\angle \mathrm{H}(\omega) = \tan^{-1} \frac{\Im m\{\mathrm{H}(\omega)\}}{\Re e\{\mathrm{H}(\omega)\}} = \tan^{-1} \frac{-\omega RC/\left(1 + (\omega RC)^2\right)}{1/(1 + (\omega RC)^2)}$$

Phase of $H(\omega)$ for RC low-pass filter

$$\mathrm{H}(\omega) = \frac{1 - \mathrm{j}\omega RC}{(1 + \mathrm{j}\omega RC)(1 - \mathrm{j}\omega RC)} = \underbrace{\frac{1 - \mathrm{j}\omega RC}{1 + (\omega RC)^2}}_{\text{Cartesian Form}}$$

$$\angle \mathrm{H}(\omega) = \tan^{-1}\frac{\Im m\{\mathrm{H}(\omega)\}}{\Re e\{\mathrm{H}(\omega)\}} = \tan^{-1}\frac{-\omega RC/\left(1 + (\omega RC)^2\right)}{1/(1 + (\omega RC)^2)}$$

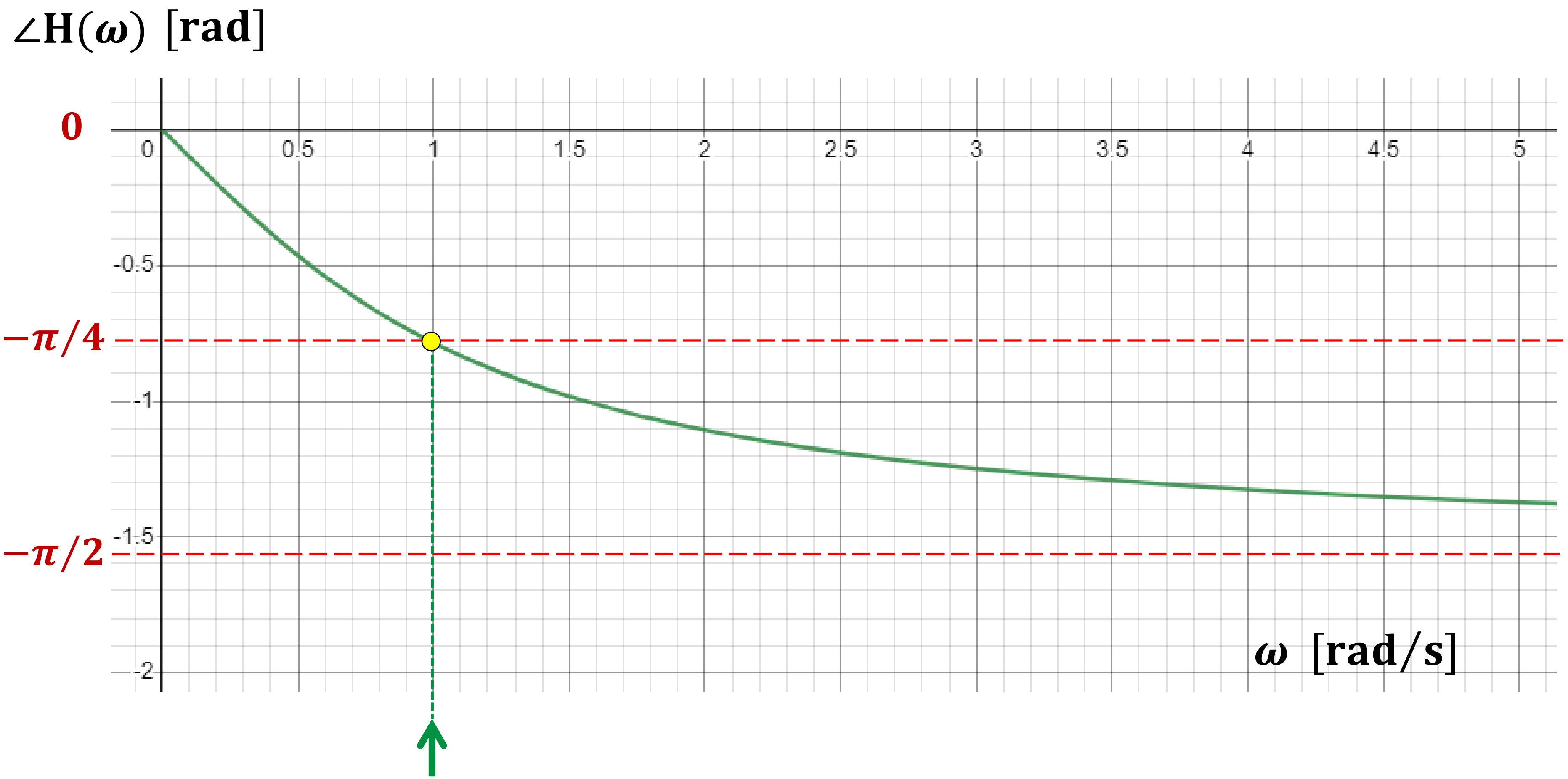
$$\boxed{\angle \mathrm{H}(\omega) = \tan^{-1}(-\omega RC) = -\tan^{-1}(\omega RC)}$$

When $\omega = \omega_C$ we have $\omega_C RC = 1$

$$\angle \mathrm{H}(\omega) = \tan^{-1}(-1) = -\frac{\pi}{4} = -45^\circ$$

Phase for RC low-pass filter

Linear scale representation



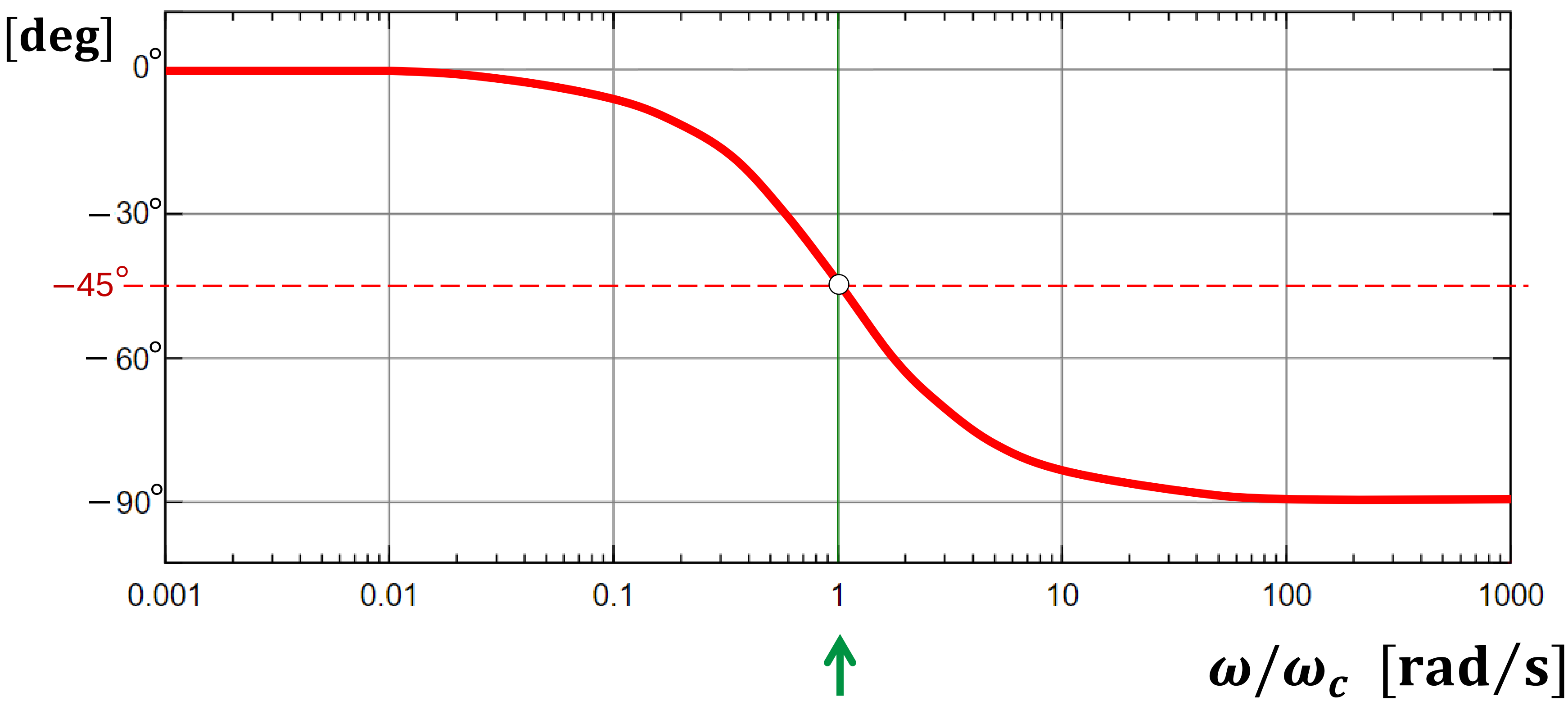
$$\omega_C = 2\pi f_C$$

Phase for RC low-pass filter

semi-log scale representation – Bode Plot for phase

NOTE: This plot is normalized so that $\omega_C = 1$

$\angle \mathrm{H}(\omega)$



$$\omega_C = 2\pi f_C$$

Example - RC low-pass filter ω_1

Consider $R = 2.5\ \mathrm{k\Omega}$ and $C = 400\ \mathrm{nF}$

$$\omega_C = 1/(RC) = \left(2.5\mathrm{k} \times 400 \times 10^{-9}\right)^{-1} = 10^3\ \ \mathrm{rad/s}$$

$f = 60\mathrm{Hz}$

$$\omega_1 \approx 377\ \mathrm{rad/s}$$

Example - RC low-pass filter ω_1

Consider $R = 2.5\ \mathrm{k\Omega}$ **and** $C = 400\ \mathrm{nF}$

$$\omega_C = 1/(RC) = \left(2.5\mathrm{k} \times 400 \times 10^{-9}\right)^{-1} = 10^3\ \mathrm{rad/s}$$

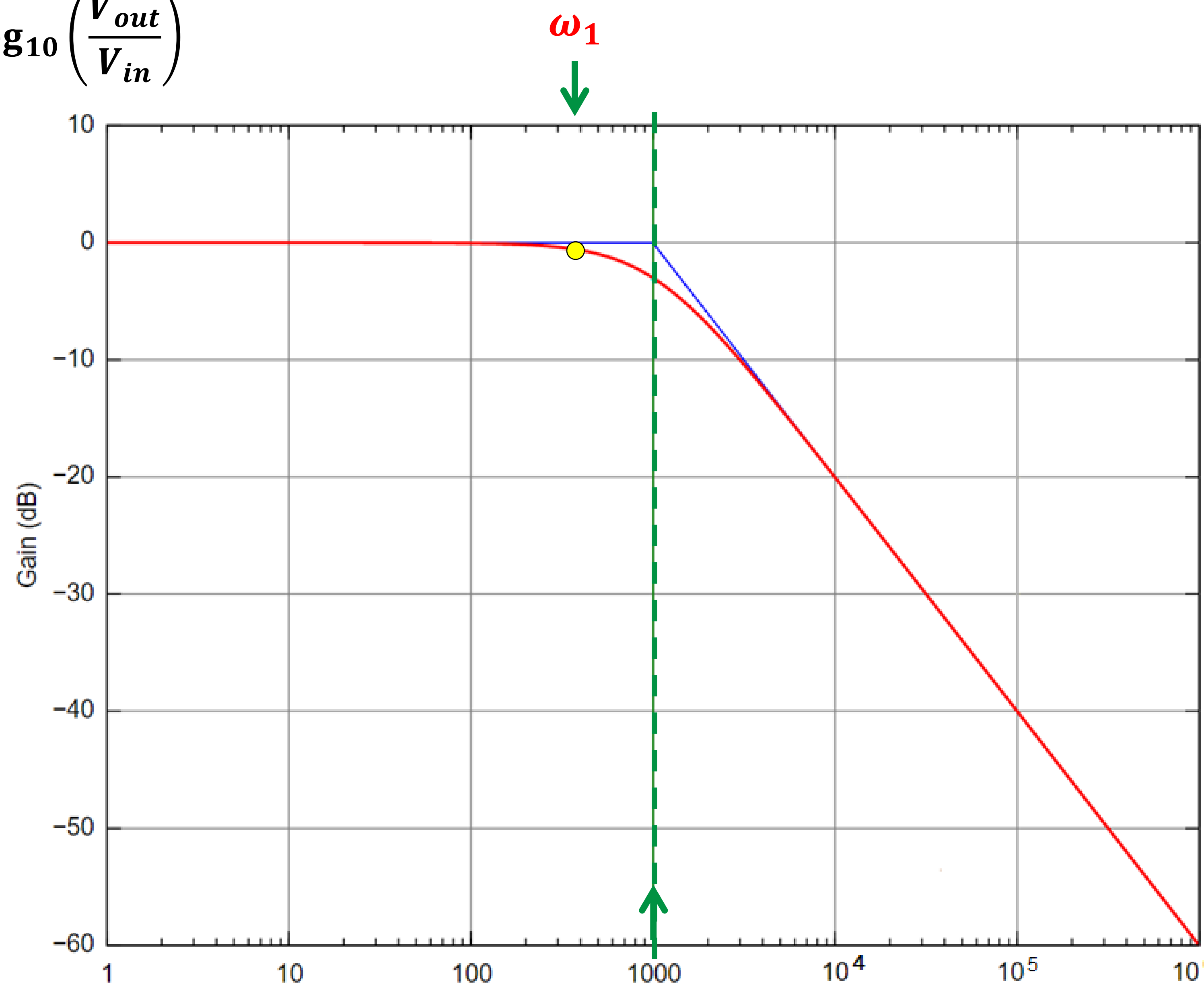
$f = 60\mathrm{Hz}$ $\qquad \omega_1 \approx 377\ \mathrm{rad/s}$

$$\omega_1 RC = 2\pi \times 60 \times 2.5\mathrm{k} \times 400 \times 10^{-9} \approx 377.0$$

$$|\mathrm{H}(\omega_1)| = \frac{1}{\sqrt{1 + (\omega_1 RC)^2}} = 0.9357$$

$$|\mathrm{H}(\omega_1)|_{\mathrm{dB}} = 20\log_{10}(0.9357) = -0.5772\ \mathrm{dB}$$

$$20\log_{10}\left(\frac{V_{out}}{V_{in}}\right)$$



$$\omega_C = 2\pi f_C$$

Example - RC low-pass filter ω_2

Consider $R = 2.5\ \mathrm{k\Omega}$ and $C = 400\ \mathrm{nF}$

$$\omega_C = 1/(RC) = \left(2.5\mathrm{k} \times 400 \times 10^{-9}\right)^{-1} = 10^3\ \mathrm{rad/s}$$

$f = 160\mathrm{Hz}$

$$\omega_2 = 1005.3\ \mathrm{rad/s}$$

Example - RC low-pass filter ω_2

Consider $R = 2.5\ \mathrm{k\Omega}$ and $C = 400\ \mathrm{nF}$

$$\omega_C = 1/(RC) = \left(2.5\mathrm{k} \times 400 \times 10^{-9}\right)^{-1} = 10^3\ \mathrm{rad/s}$$

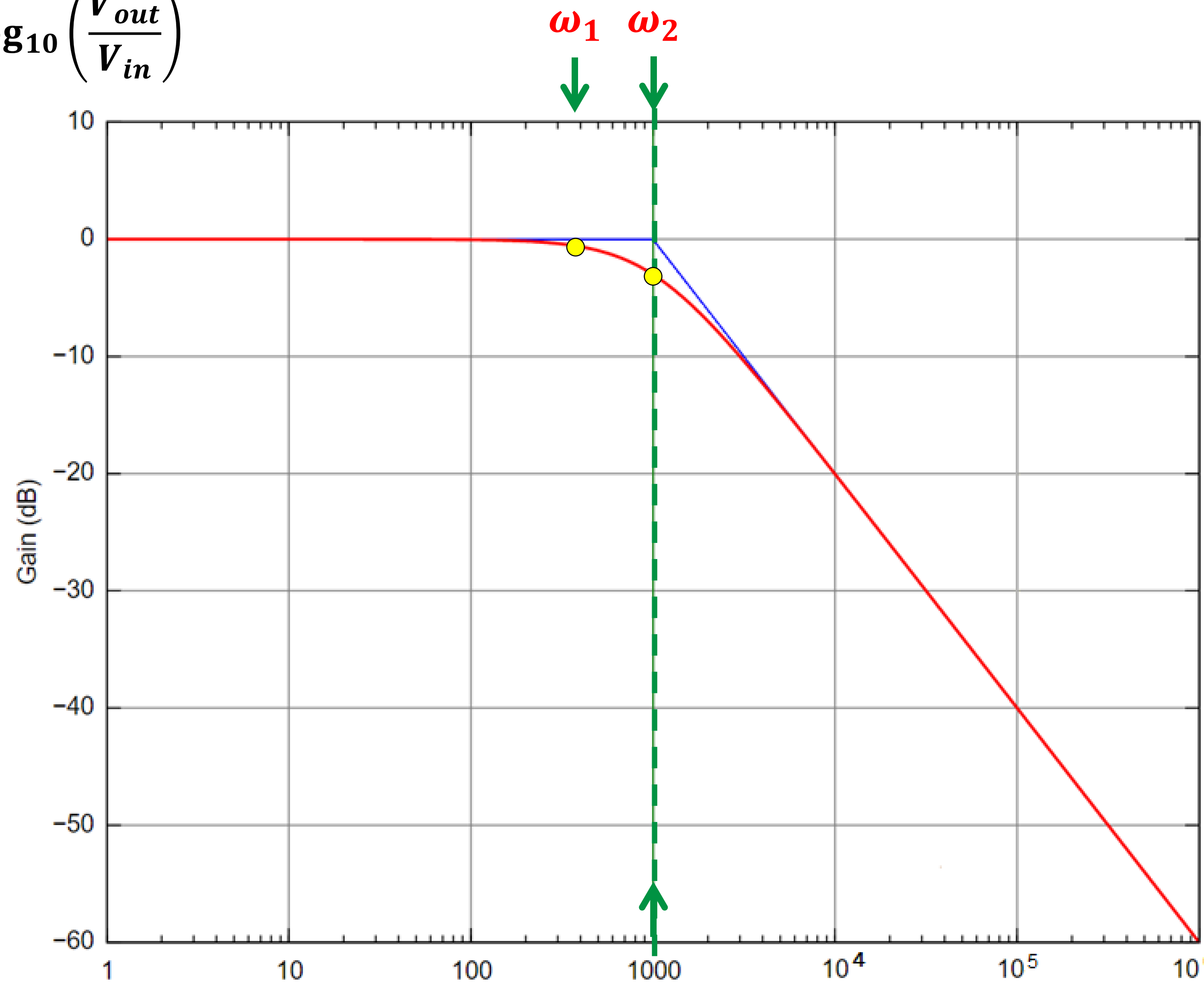
$f = 160\mathrm{Hz}$ $\qquad$ $\omega_2 = 1005.3\ \mathrm{rad/s}$

$$\omega_2 RC = 2\pi \times 160 \times 2.5\mathrm{k} \times 400 \times 10^{-9} \approx 1.0$$

$$|\mathrm{H}(\omega_2)| = \frac{1}{\sqrt{1 + (\omega_2 RC)^2}} = 0.7052$$

$$|\mathrm{H}(\omega_2)|_{\mathrm{dB}} = 20\log_{10}(0.7052) = -3.03\overline{3}\ \mathrm{dB}$$

$$20\log_{10}\left(\frac{V_{out}}{V_{in}}\right)$$



$$\omega_C = 2\pi f_C$$

Example - RC low-pass filter ω_3

Consider $R = 2.5\ \text{k}\Omega$ and $C = 400\ \text{nF}$

$$\omega_C = 1/(RC) = \left(2.5\text{k} \times 400 \times 10^{-9}\right)^{-1} = 10^3\ \text{rad/s}$$

$f = 16\ \text{kHz}$

$\omega_3 = 100{,}530\ \text{rad/s}$

Example - RC low-pass filter ω_3

Consider $R = 2.5\ \mathrm{k\Omega}$ and $C = 400\ \mathrm{nF}$

$$\omega_C = 1/(RC) = \left(2.5\mathrm{k} \times 400 \times 10^{-9}\right)^{-1} = 10^3\ \mathrm{rad/s}$$

$f = 16\ \mathrm{kHz}$

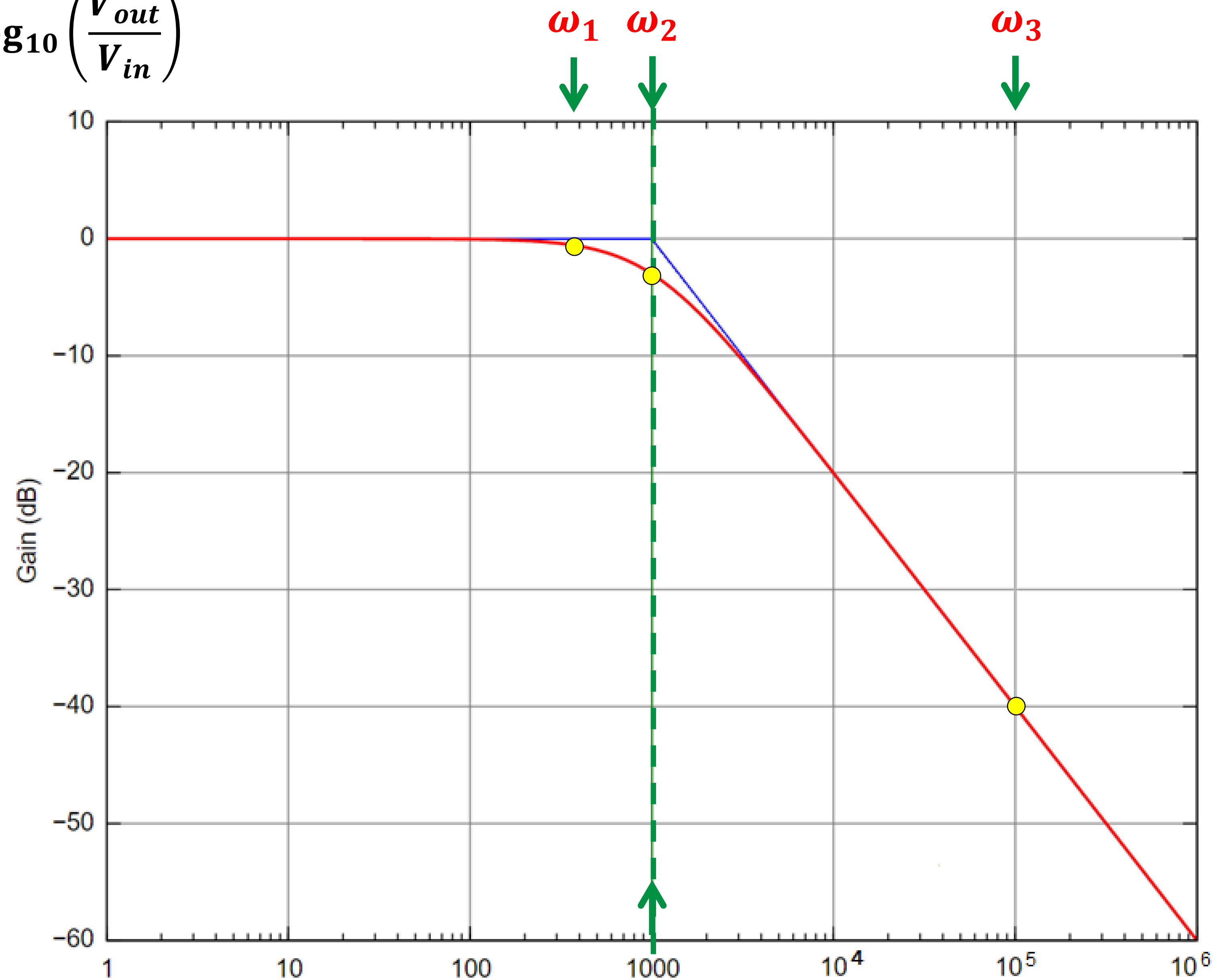
$$\omega_3 = 100{,}530\ \mathrm{rad/s}$$

$$\omega_3 RC = 2\pi \times 16\mathrm{k} \times 2.5\mathrm{k} \times 400 \times 10^{-9} \approx 100.53$$

$$|\mathrm{H}(\omega_3)| = \frac{1}{\sqrt{1 + (\omega_3 RC)^2}} = 0.0099$$

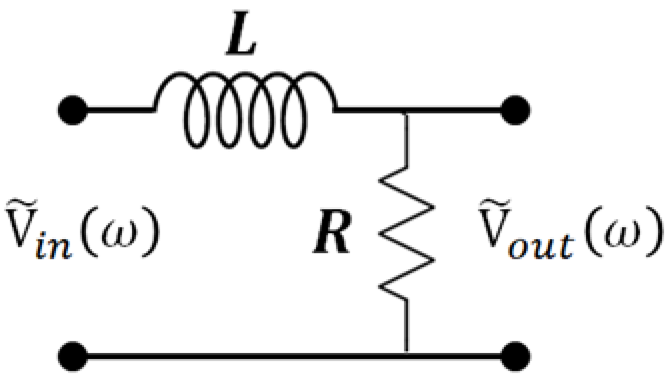
$$|\mathrm{H}(\omega_3)|_{\mathrm{dB}} = 20 \log_{10}(0.0099) = -40.046\ \mathrm{dB}$$

$$20\log_{10}\left(\frac{V_{out}}{V_{in}}\right)$$

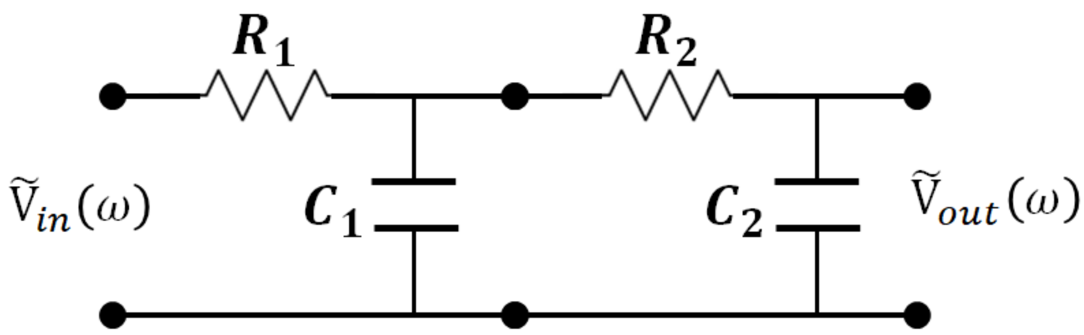


$$\omega_C = 2\pi f_C$$

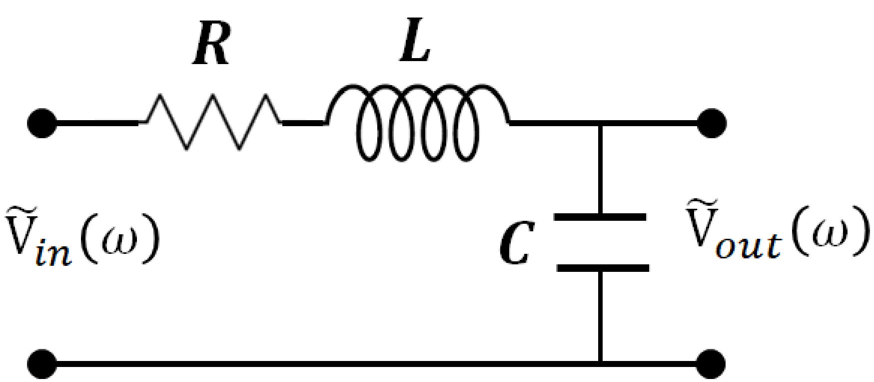
Other Low-Pass Passive filter configurations



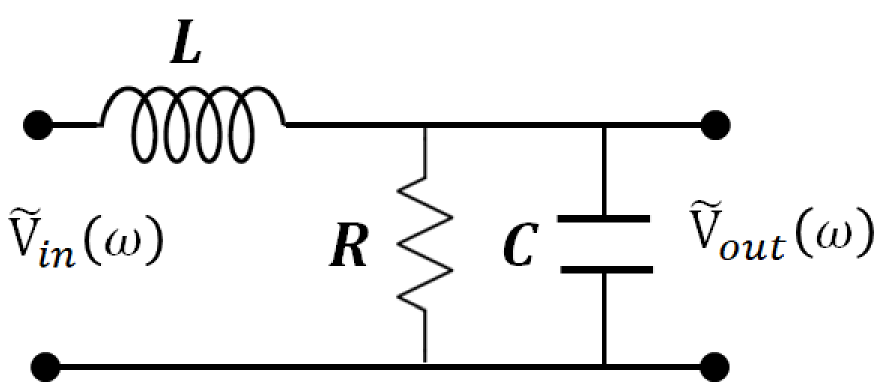
RL filter (1st order)



RC filter (2nd order)



RLC filter (2nd order)

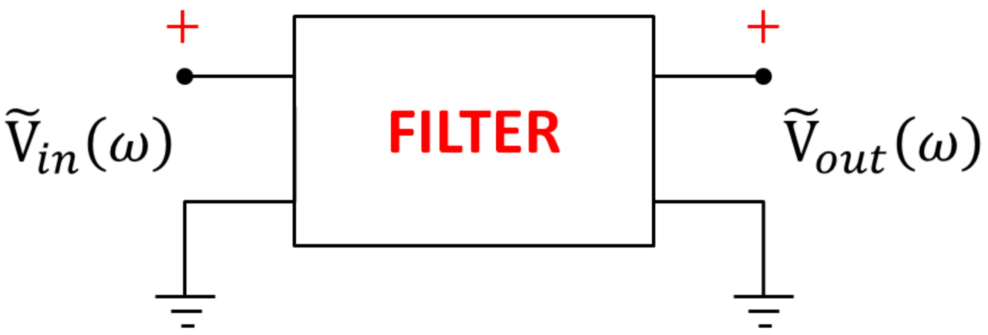


RLC filter (2nd order)

Sinusoidal input signal

$$V_{in}(t) = V_m \cos(\omega t + \theta_V)$$

$$H(\omega) = |H(\omega)| \angle H(\omega)$$



$$\tilde{V}_{out}(\omega) = H(\omega)\,\tilde{V}_{in}(\omega)$$

$$V_{out}(t) = |H(\omega)| V_m \cos(\omega t + \theta_V + \angle H(\omega))$$

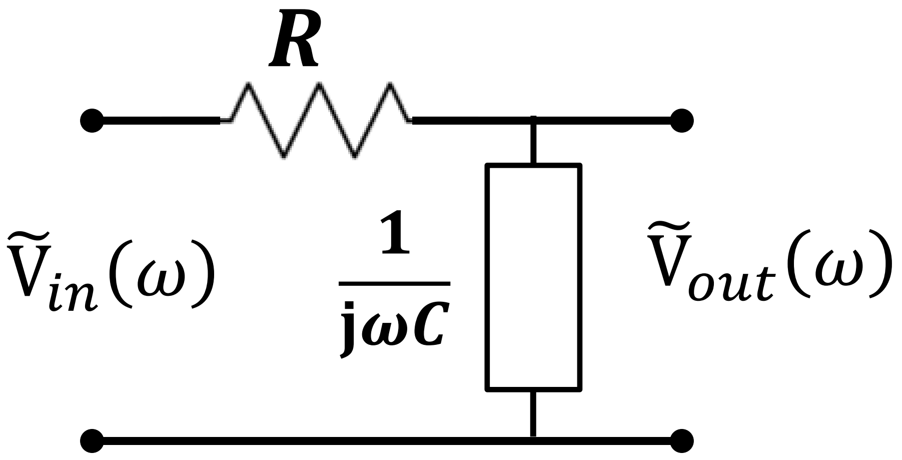
Example – Low Pass RC filter

$$V_{in}(t) = 2\cos(3000t)$$

Find $V_{out}(t)$

$$\mathrm{H}(\omega) = \frac{1}{1 + \mathrm{j}\omega RC}$$

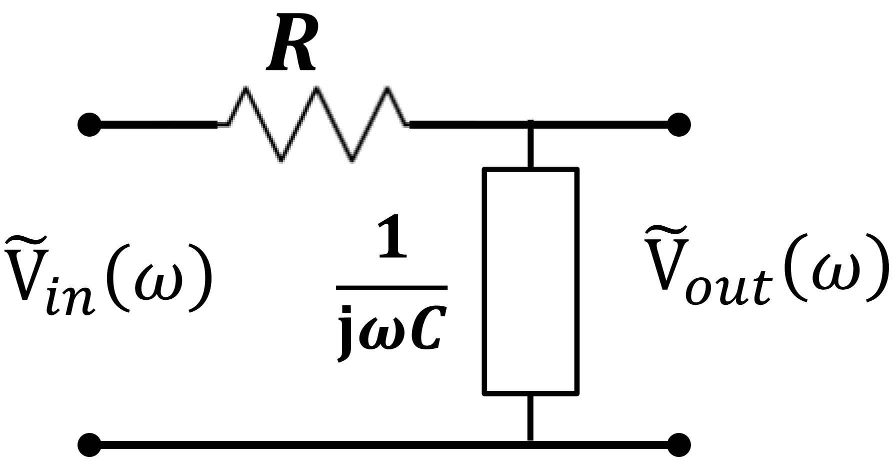
Let $R = 1\mathrm{k}\Omega$ and $C = 1\mu\mathrm{F}$.



Example – Low Pass RC filter

$V_{in}(t) = 2\cos(3000t)$

Find $V_{out}(t)$



$$H(\omega) = \frac{1}{1 + j\omega RC}$$

Let $R = 1\text{k}\Omega$ and $C = 1\mu\text{F}$.

$$\omega RC = 3000 \times 1k \times 1\mu = 3$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (3)^2}} = \frac{1}{\sqrt{10}}\ \text{V}$$

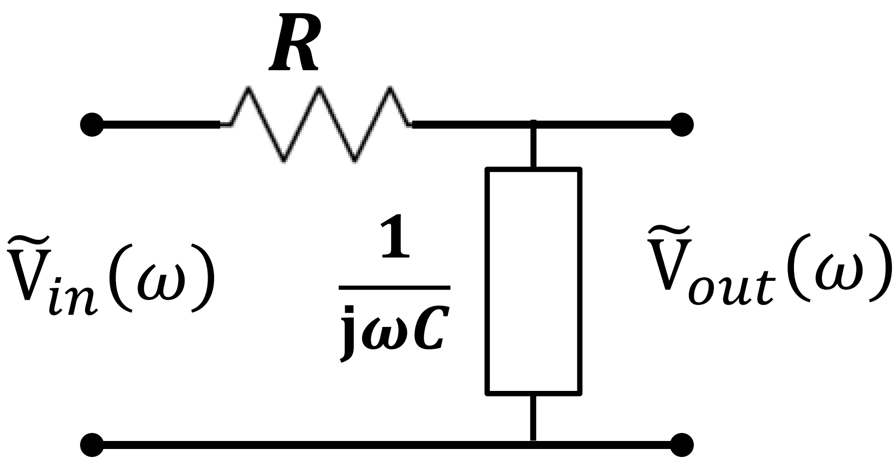
$$\angle H(\omega) = -\tan^{-1}(\omega RC)$$

$$\angle H(\omega) = -\tan^{-1}(3) = -1.249\ \text{rad}$$

Example – Low Pass RC filter

$V_{in}(t) = 2\cos(3000t)$

Find $V_{out}(t)$



$$H(\omega) = \frac{1}{1 + j\omega RC}$$

Let $R = 1\text{k}\Omega$ and $C = 1\mu\text{F}$.

$$|H(\omega)| = \frac{1}{\sqrt{1 + (3)^2}} = \frac{1}{\sqrt{10}} \text{ V}$$

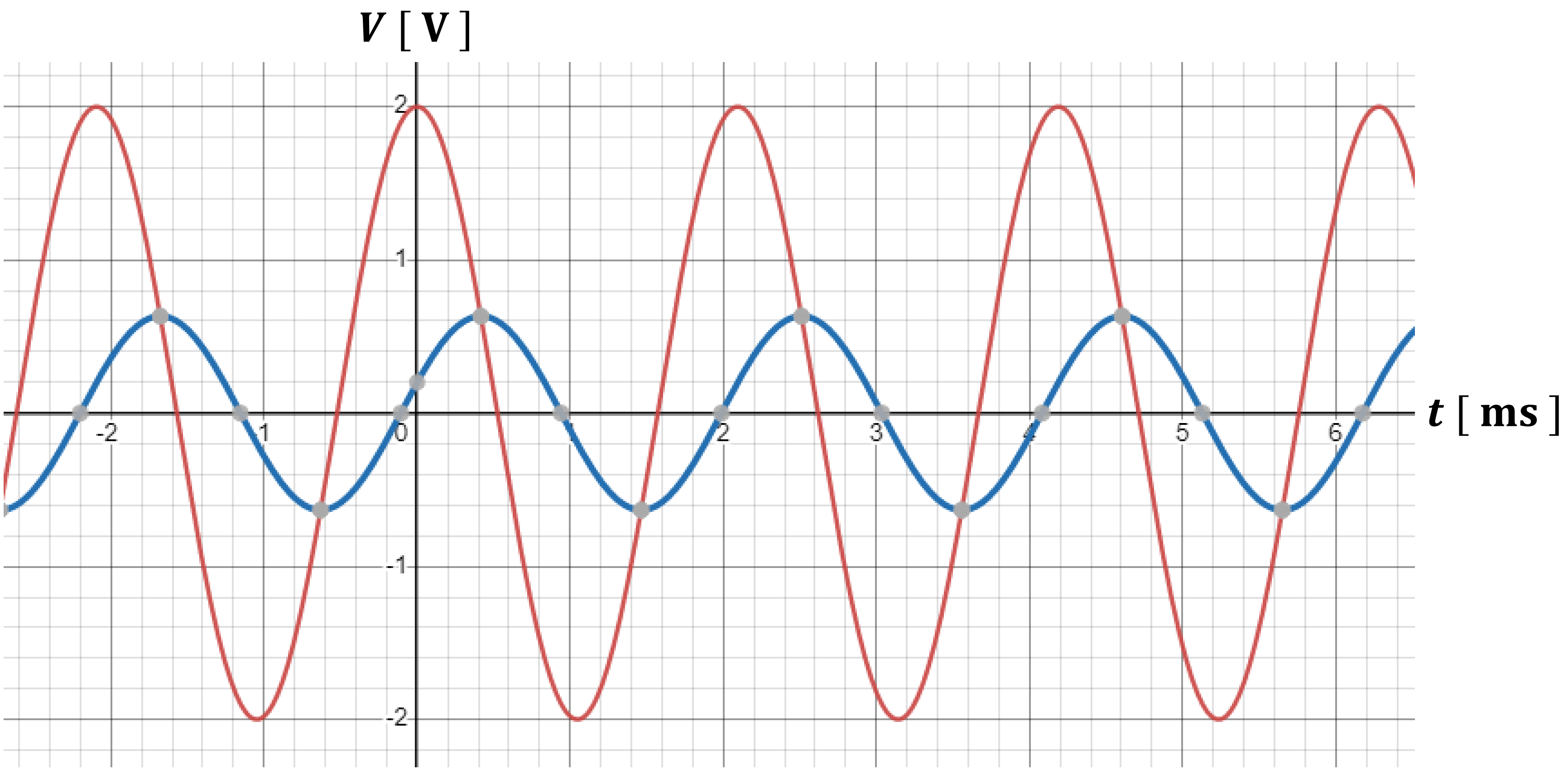
$$\angle H(\omega) = -\tan^{-1}(3) = -1.249 \text{ rad}$$

$$V_{out}(t) = |H(\omega)| \times 2\cos(3000t + \angle H(\omega))$$

$$V_{out}(t) = \frac{2}{\sqrt{10}} \cos(3000t - 1.249)$$

Valid for sinusoidal signals

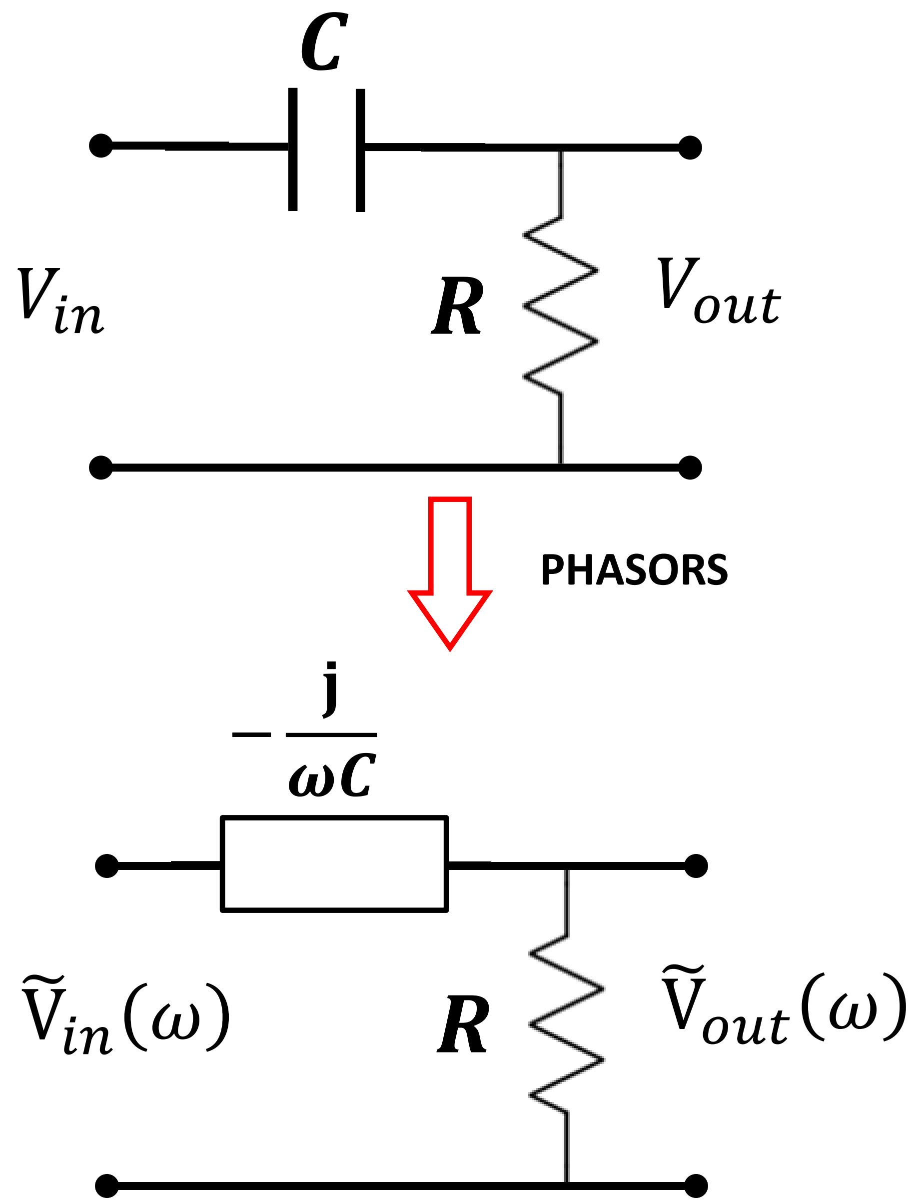
$$V_{in}(t) = 2\cos(3000t) \qquad V_{out}(t) = \frac{2}{\sqrt{10}}\cos(3000t - 1.249)$$

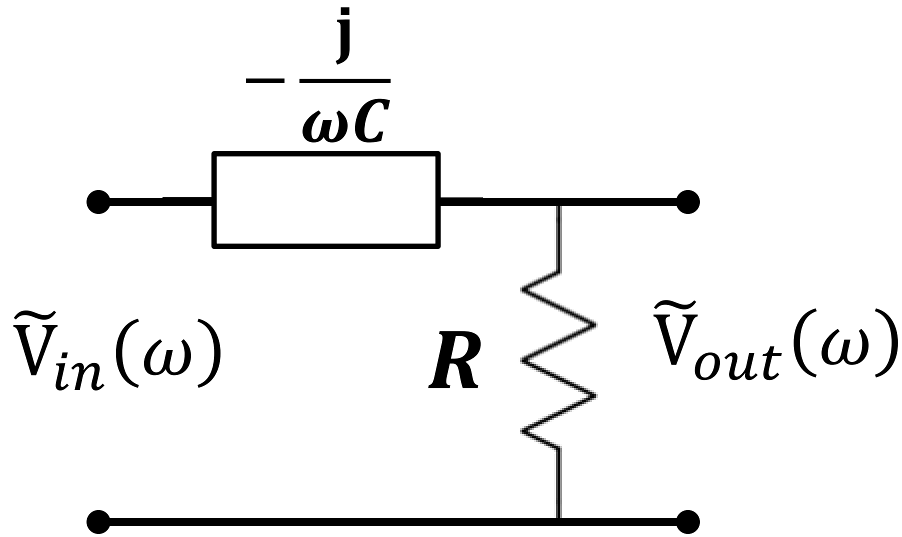


The filter introduces a time delay for the output signal

High Pass RC filter

RC filter
(1st order)





Let the input be a phasor of the form

$$\tilde{V}_{in}(\omega) = V_I \angle 0°$$

$$\tilde{V}_{out}(\omega) = V_I \angle 0° \frac{R}{R + 1/j\omega C} = \underbrace{V_I \angle 0°}_{\tilde{V}_{in}(\omega)} \frac{j\omega RC}{1 + j\omega RC}$$

$$\boxed{\frac{\tilde{V}_{out}(\omega)}{\tilde{V}_{in}(\omega)} = H(\omega) = \frac{j\omega RC}{1 + j\omega RC}}$$

Transfer Function

$$\mathrm{H}(\omega) = \frac{\mathrm{j}\omega RC}{1 + \mathrm{j}\omega RC}$$

$$\mathrm{H}(\omega) = \frac{\mathrm{j}\omega RC(1 - \mathrm{j}\omega RC)}{(1 + \mathrm{j}\omega RC)(1 - \mathrm{j}\omega RC)} = \frac{(\omega RC)^2 + \mathrm{j}\omega RC}{1 + (\omega RC)^2}$$

Cartesian Form

Magnitude

$$|\mathrm{H}(\omega)| = \frac{|\mathrm{j}\omega RC|}{|1 + \mathrm{j}\omega RC|}$$

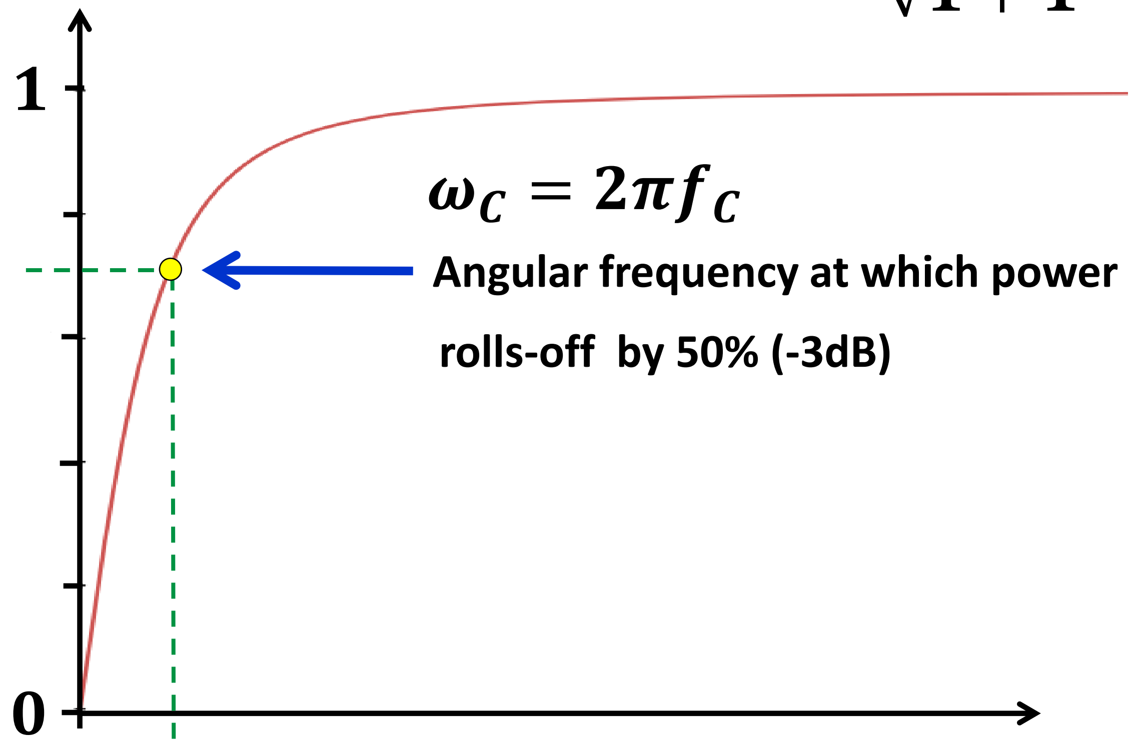
$$|\mathrm{H}(\omega)| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$

Magnitude of $H(\omega)$ for RC high-pass filter

$$|\mathrm{H}(\omega)| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$

$$\omega_C RC = 1 \rightarrow |\mathrm{H}(\omega_C)| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

$|\mathrm{H}(\omega)|$



$\frac{1}{\sqrt{2}} \approx 0.707$

$\omega_C = 2\pi f_C$

Angular frequency at which power $P \propto |H(\omega)|^2$ rolls-off by 50% (-3dB)

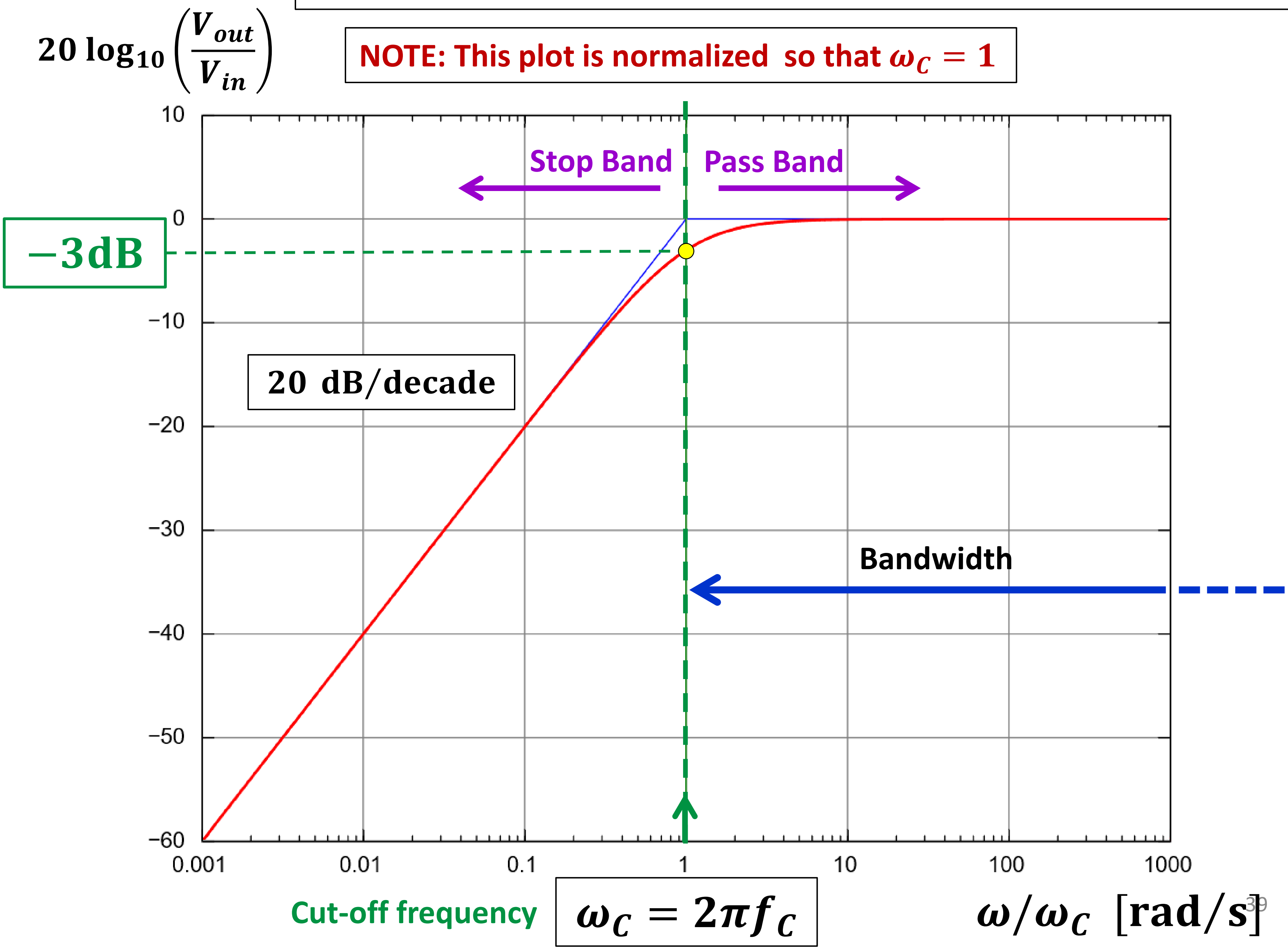
$\tau = RC$

$\omega_C = \tau^{-1}$

ω

log-decibel representation – Bode Plot for magnitude

NOTE: This plot is normalized so that $\omega_C = 1$



Phase of $H(\omega)$ for RC high-pass filter

$$\mathrm{H}(\omega) = \frac{\mathrm{j}\omega RC(1 - \mathrm{j}\omega RC)}{(1 + \mathrm{j}\omega RC)(1 - \mathrm{j}\omega RC)} = \underbrace{\frac{(\omega RC)^2 + \mathrm{j}\omega RC}{1 + (\omega RC)^2}}_{\text{Cartesian Form}}$$

$$\angle\mathrm{H}(\omega) = \tan^{-1}\frac{\Im m\{\mathrm{H}(\omega)\}}{\Re e\{\mathrm{H}(\omega)\}} = \tan^{-1}\frac{\omega RC/\left(1 + (\omega RC)^2\right)}{(\omega RC)^2/(1 + (\omega RC)^2)}$$

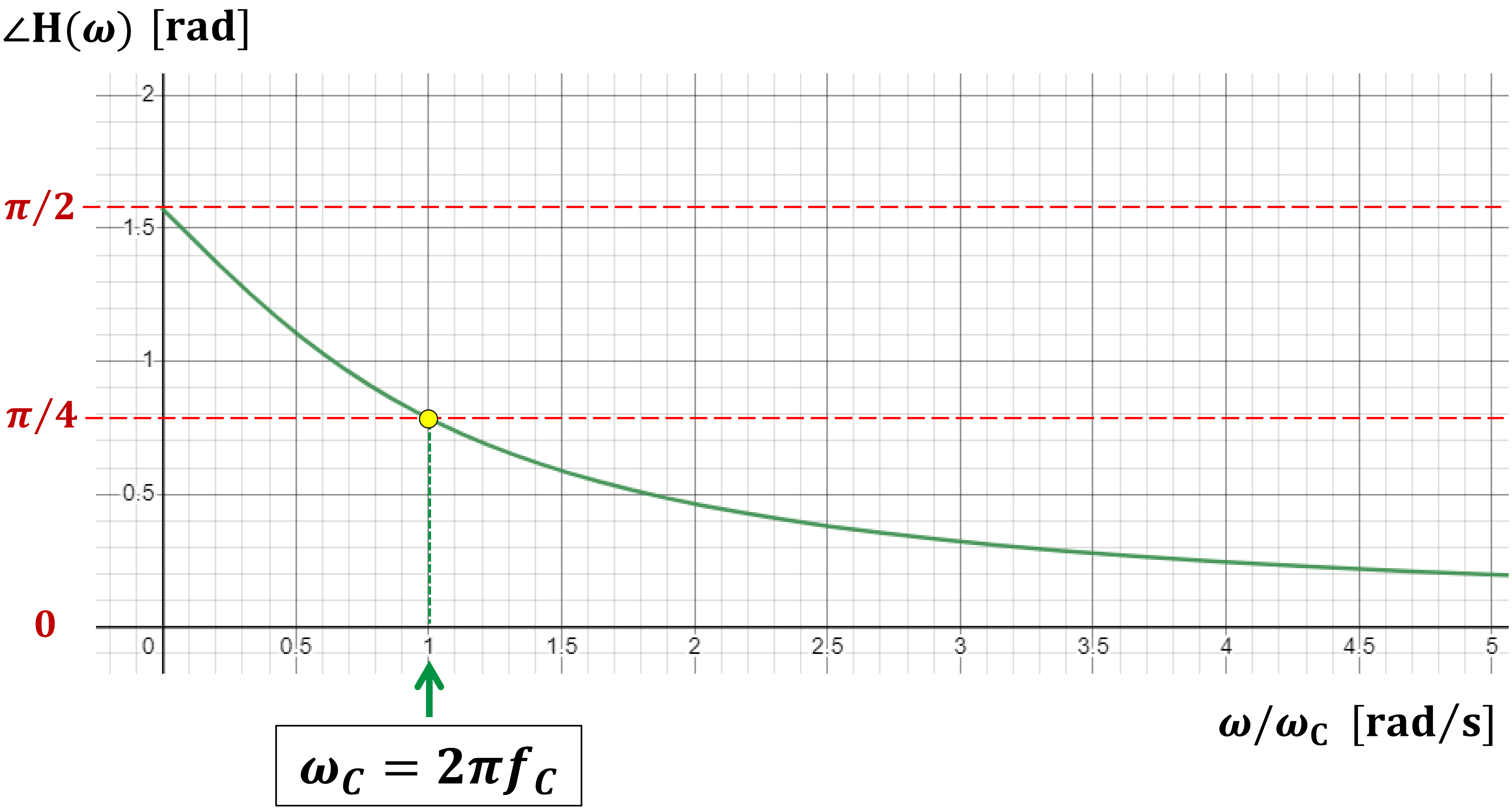
$$\angle\mathrm{H}(\omega) = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

When $\omega = \omega_C$ we have $\omega_C RC = 1$

$$\angle\mathrm{H}(\omega) = \tan^{-1}(1) = \frac{\pi}{4} = 45^\circ$$

Phase for RC high-pass filter

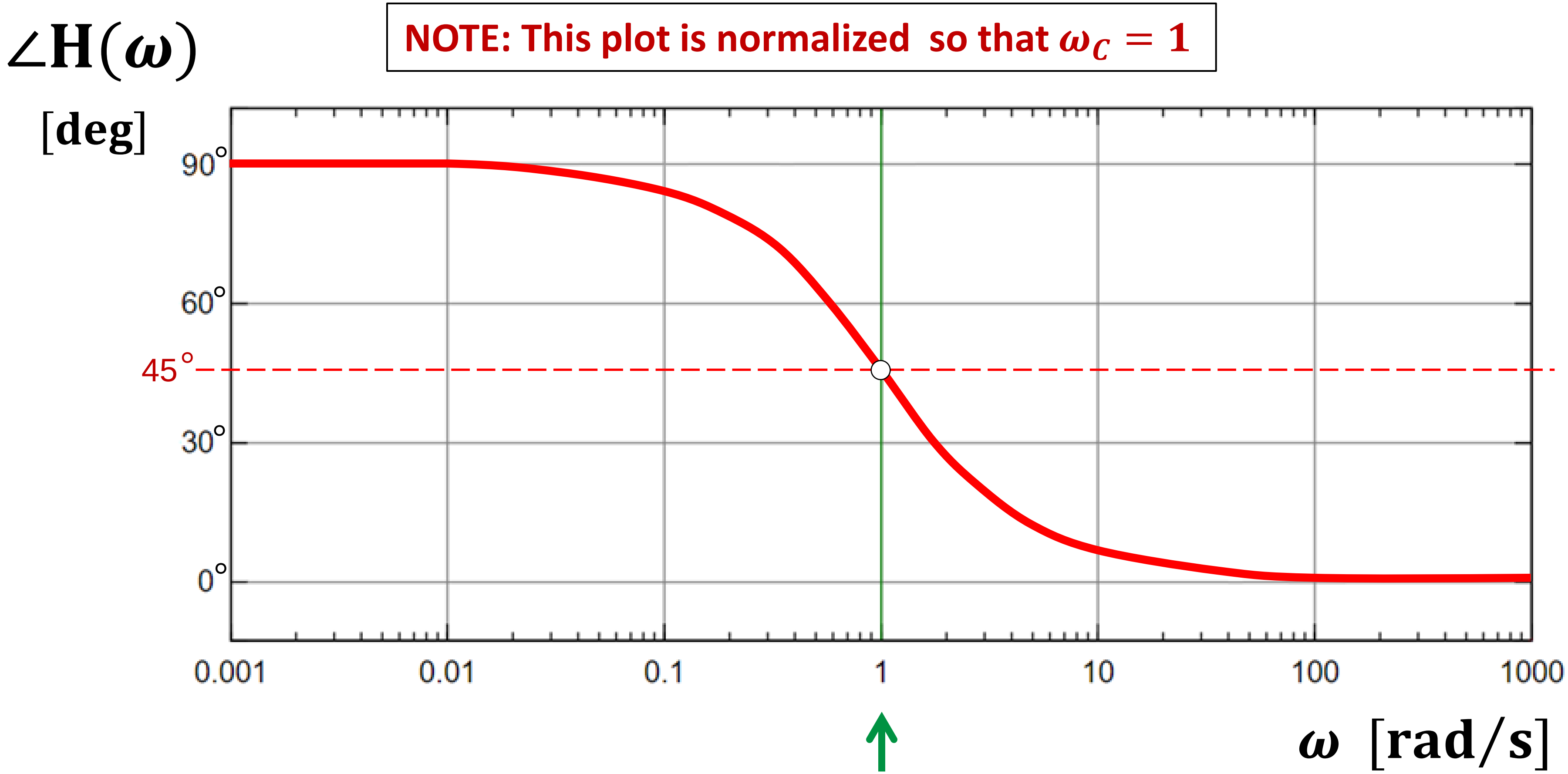
Linear scale representation



Phase for RC high-pass filter

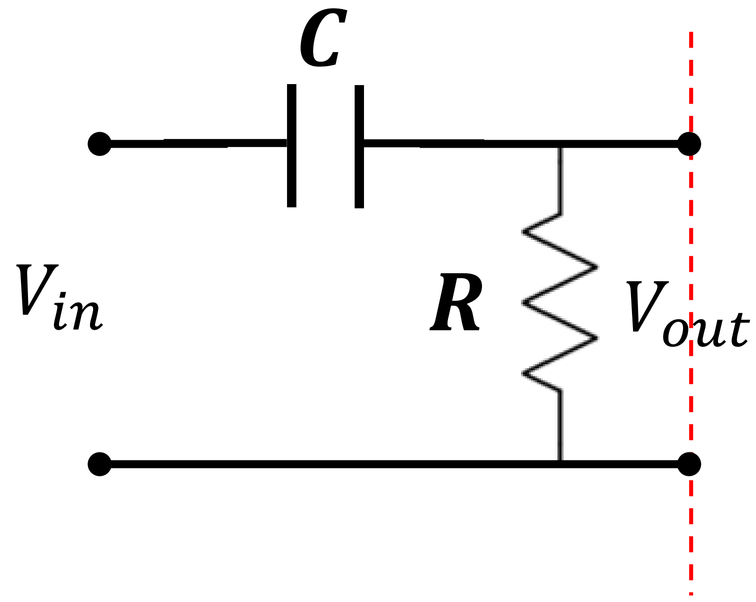
semi-log scale representation – Bode Plot for phase

NOTE: This plot is normalized so that $\omega_C = 1$



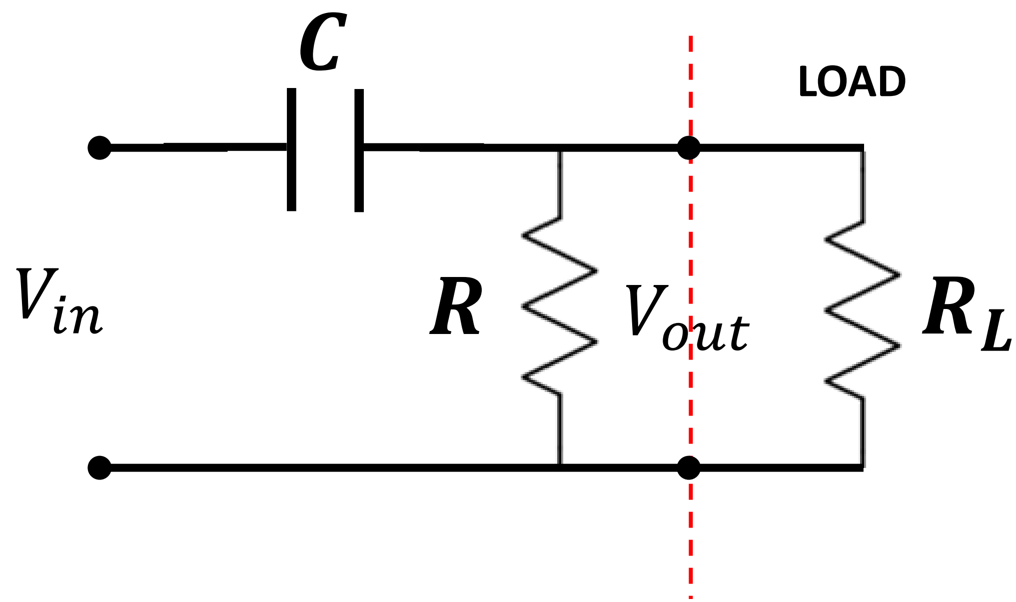
$$\omega_C = 2\pi f_C$$

Limitations of simple passive filters



Limitations of simple passive filters

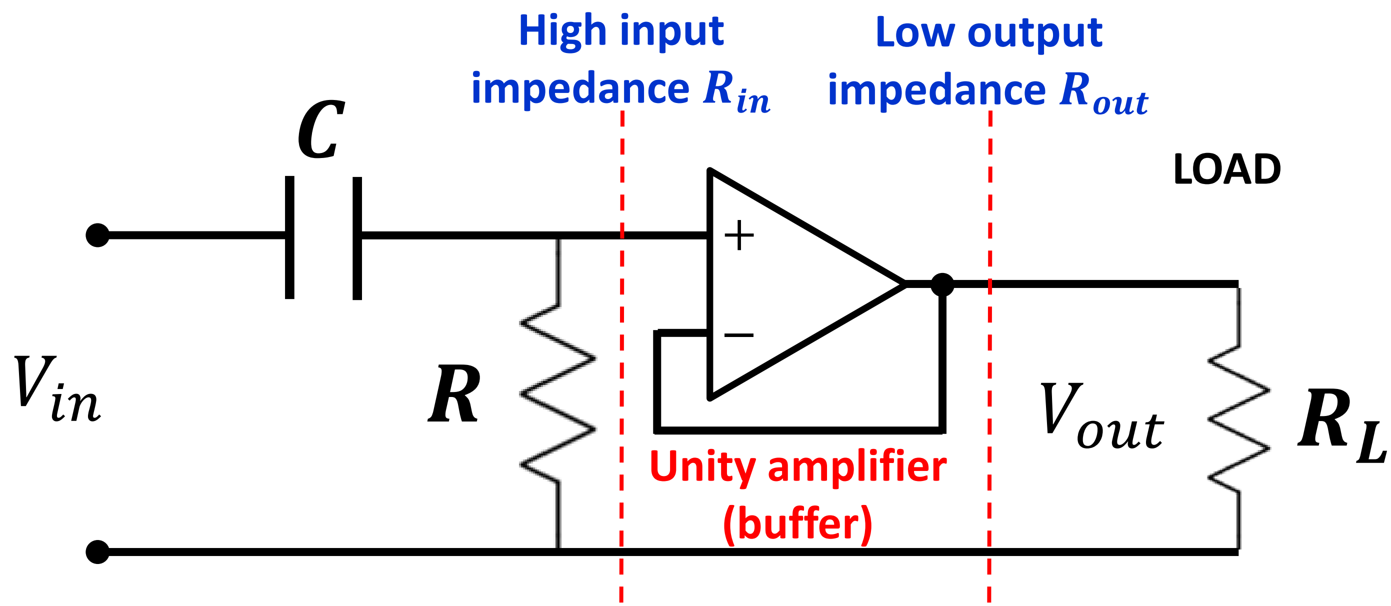
The response of a passive filter is affected by the load connected directly to it. For example, consider a high-pass RC filter:



The parallel $\mathrm{R}_{\mathrm{eff}} = R//R_L$ yields an equivalent resistance lower than either R or R_L. In particular, if connected to a small resistor R_L, the resulting cutoff frequency $\omega'_C = (R_{\mathrm{eff}}C)^{-1}$ may change considerably with respect to the original $\omega_C = (RC)^{-1}$.

Overcome limitations by using active filters

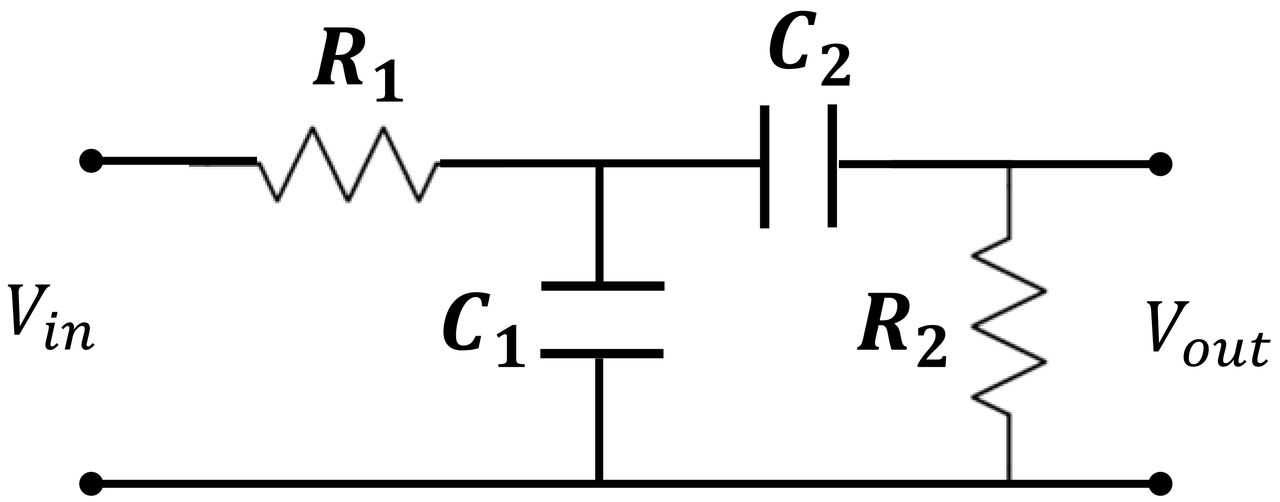
In the final part of the course, we will learn how a high input impedance operational amplifier can be used as an intermediate stage to improve the interconnection between filter and load.



The filter sees $R//R_{in} \approx R$ and is not affected. In output the voltage V_{out} drives a total resistance $R_{out} + R_L$. If R_{out} is much smaller, power is delivered mainly to the load R_L.

Band-Pass Filter

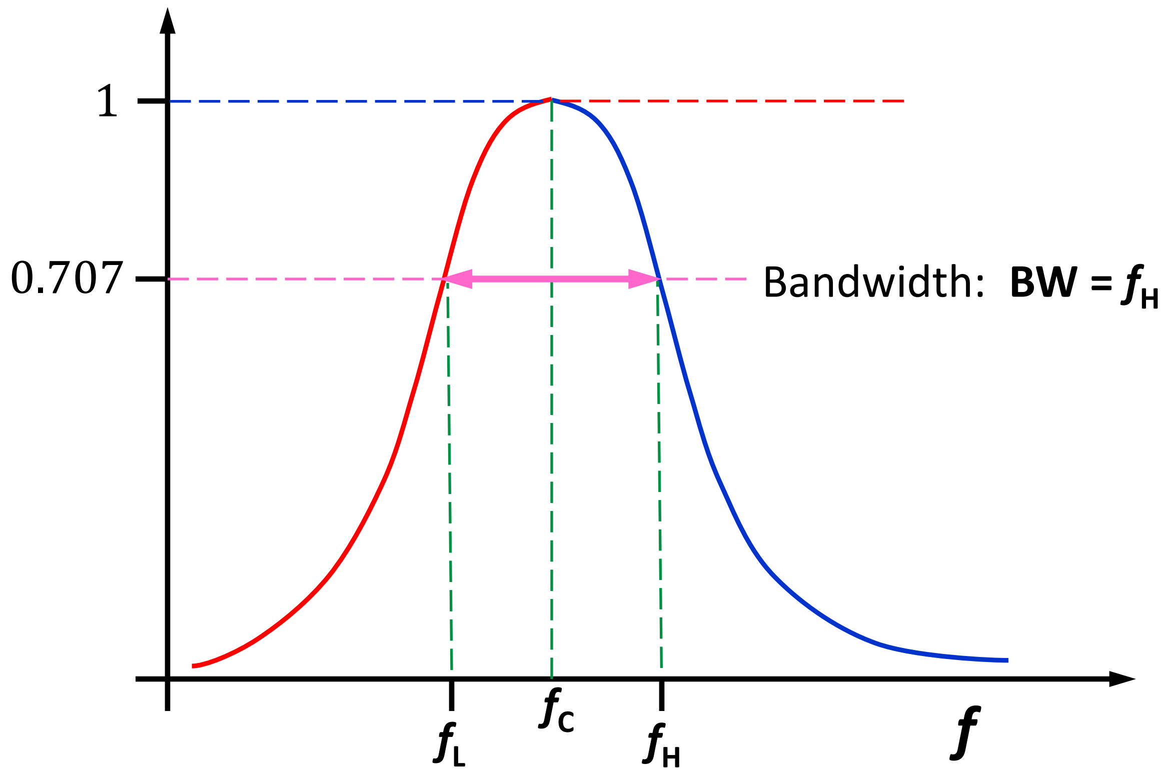
Cascade of a low pass and a high-pass filter can be designed so that $\omega_{CLP} > \omega_{CHP}$. The two filter characteristics combine, letting only an intermediate frequency band pass through.



Looking at frequency extremes, one can see that:

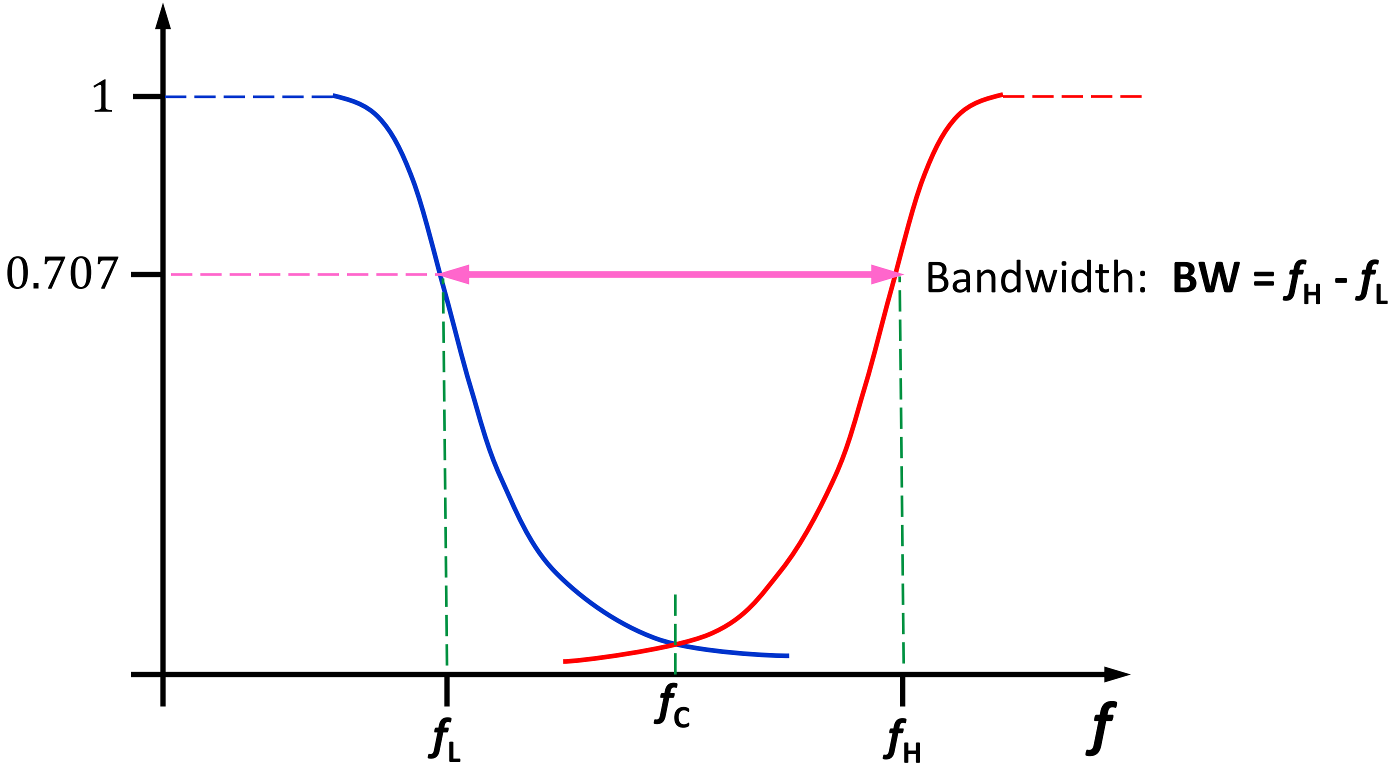
- $\omega = 0$ **capacitors behave like open circuit** $\rightarrow V_{out} = 0$
- $\omega \rightarrow \infty$ **capacitors behave like short circuit** $\rightarrow V_{out} = 0$

Band-Pass Filter



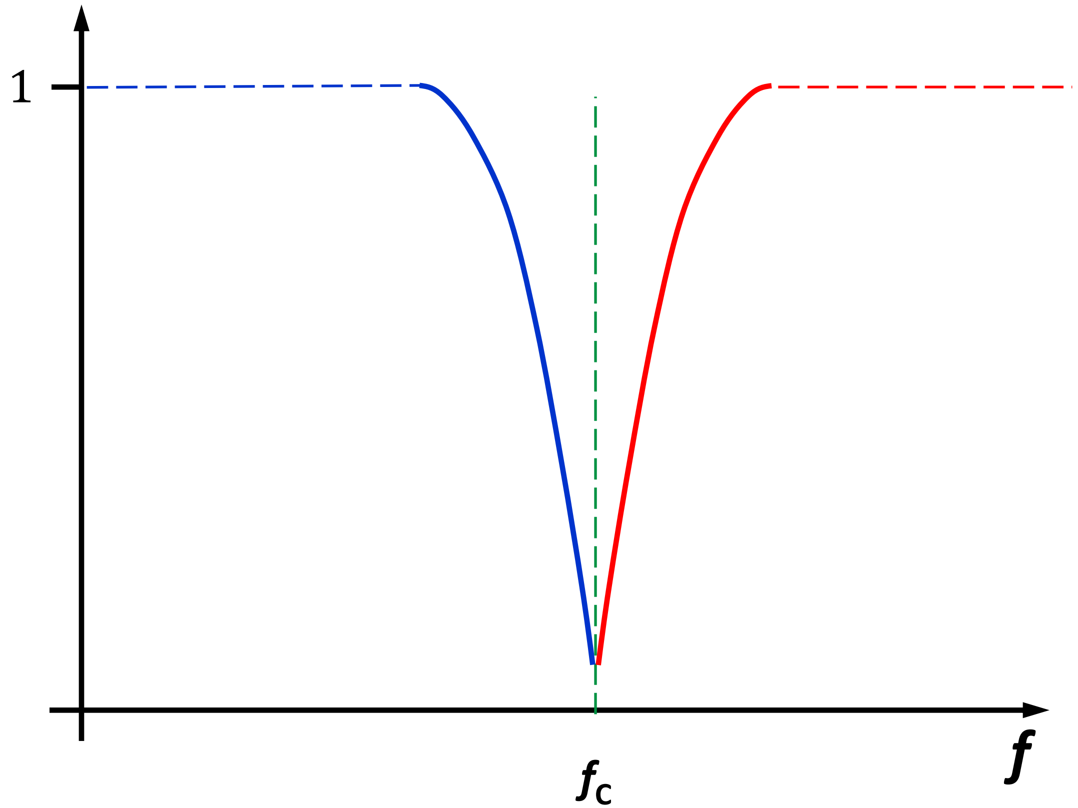
Frequencies $f < f_L$ and $f > f_H$ are strongly attenuated

Band-Stop Filter



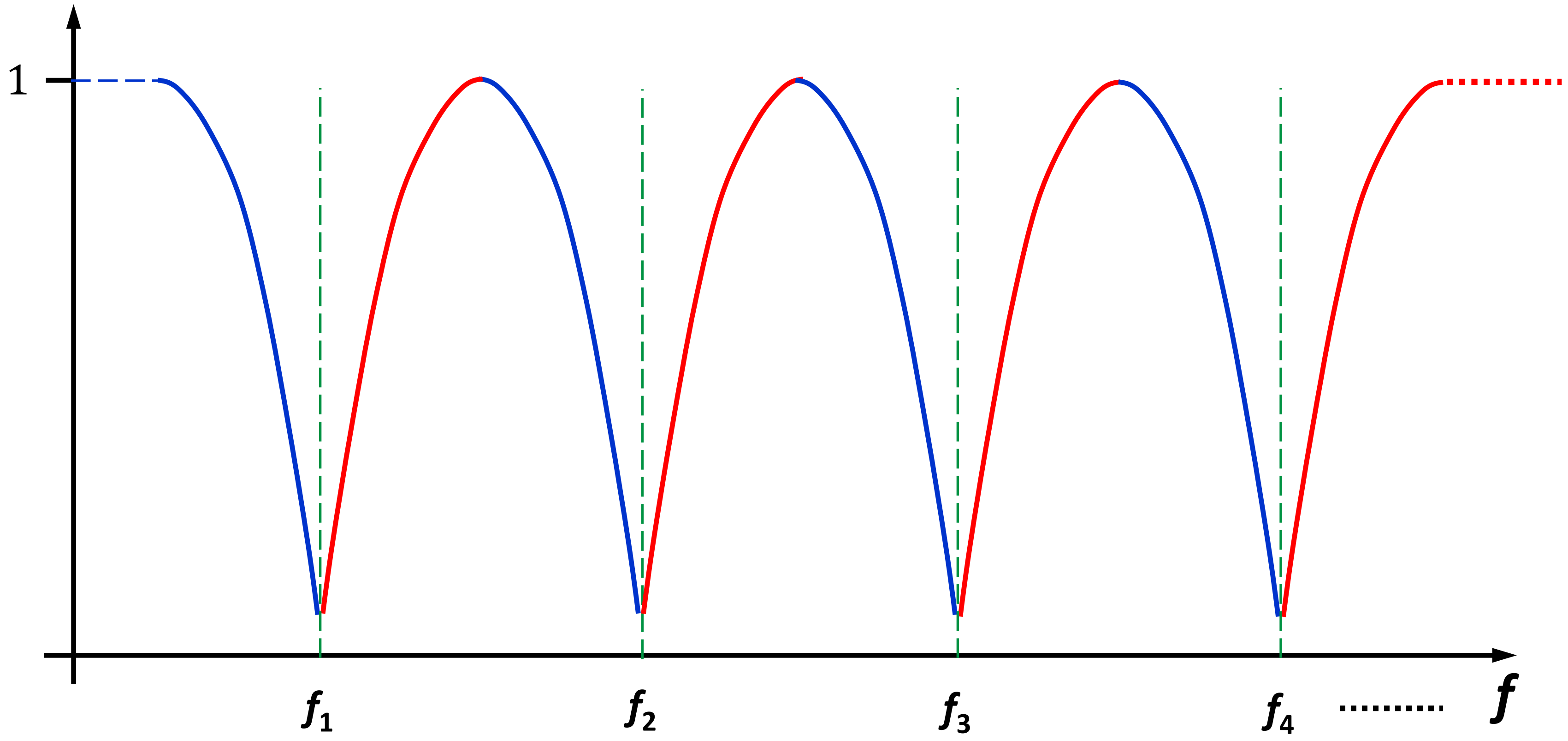
Frequencies $f_L < f < f_H$ are strongly attenuated

Notch Filter



A very narrow band-stop filter, designed to reject a specific frequency, is called a notch filter.

Comb Filter



The comb filter consists of a series of regularly spaced notches and peaks (also called *teeth*).